EXERCISE 4.1 PAGE NO: 4.6

1. Find the principal value of the following:

$(i)\ sin^{-1}(-\sqrt{\frac{3}{2}})$

$(ii)\ sin^{-1}(\cos\frac{2\pi}{3})$

$(iii)\ \sin^{-1}(\frac{\sqrt{3}-1}{2\sqrt{2}})$

$(iv)\ \sin^{-1}(\frac{\sqrt{3}+1}{2\sqrt{2}})$

$(v)\ \sin^{-1}(\cos\frac{3\pi}{4})$

$(vi)\ \sin^{-1}(\tan\frac{5\pi}{4})$

Solution:

$(i) Let \sin^{-1}(\frac{-\sqrt{3}}{2}) = y$

$Then\ siny = (\frac{-\sqrt{3}}{2})$

$= -\sin(\frac{\pi}{3})$

$= sin(-\frac{\pi}{3})$

$We\ know\ that\ the\ principal\ value\ of\ \sin^{-1}\ is\ \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

$And - sin\frac{\pi}{3} = sin(\frac{-\pi}{3})$

$Therefore\ principal\ value\ of \sin^{-1}(\frac{-\sqrt{3}}{2}) = \frac{-\pi}{3}$

$(ii) Let \sin^{-1}(\cos\frac{2\pi}{3}) = y$

$Then\ \ siny = \cos(\frac{2\pi}{3})$

$= -\sin(\frac{\pi}{2}+\frac{\pi}{6})$

$= -sin(\frac{\pi}{6})$

$We\ know\ that\ the\ principal\ value\ of\ \sin^{-1}\ is\ \left[\frac{-\pi}{2},\frac{\pi}{2}\right]$

$And\ \ -sin(\frac{\pi}{6}) = cos(\frac{2\pi}{3})$

$Therefore\ principal\ value\ of\ \sin^{-1}(\cos\frac{2\pi}{3})\ is \frac{-\pi}{6}$

(iii) Given functions can be written as

$$\sin^{-1}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right) = \sin^{-1}\left(\frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}\right)$$

Taking 1/√2 as common from the above equation we get,

$$= \sin^{-1}\left(\frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}} - \frac{1}{2}\times\frac{1}{\sqrt{2}}\right)$$

Taking √3/2 as common, and 1/√2 from the above equation we get,

$$= \sin^{-1}\left(\frac{\sqrt{3}}{2}\times\sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2} - \frac{1}{\sqrt{2}}\times\sqrt{1-\left(\frac{\sqrt{3}}{2}\right)^2}\right)$$

On simplifying, we get

$$= \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

By substituting the values,

$$= \frac{\pi}{3} - \frac{\pi}{4}$$

Taking LCM and cross multiplying we get,

$$= \frac{\pi}{12}$$

(iv) The given question can be written as

$$\sin^{-1}\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right) = \sin^{-1}\left(\frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}\right)$$

Taking 1/√2 as common from the above equation we get

$$= \sin^{-1}\left(\frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} + \frac{1}{2} \times \frac{1}{\sqrt{2}}\right)$$

Taking √3/2 as common, and 1/√2 from the above equation we get,

$$= \sin^{-1}\left(\frac{\sqrt{3}}{2} \times \sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2} + \frac{1}{\sqrt{2}} \times \sqrt{1-\left(\frac{\sqrt{3}}{2}\right)^2}\right)$$

On simplifying we get,

$$= \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) + \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

By substituting the corresponding values we get

$$= \frac{\pi}{3} + \frac{\pi}{4}$$

$$= \frac{7\pi}{12}$$

(v) Let

$\sin^{-1}\left(\cos\frac{3\pi}{4}\right) = y$

Then above equation can be written as

$\sin y = \cos\frac{3\pi}{4} = -\sin\left(\pi - \frac{3\pi}{4}\right) = -\sin(\frac{\pi}{4})$

We know that the principal value of $\sin^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Therefore above equation becomes,

$-\sin\left(\frac{\pi}{4}\right) = \cos\frac{3\pi}{4}$

Therefore the principal value of $\sin^{-1}\left(\cos\frac{3\pi}{4}\right)$ is $-\frac{\pi}{4}$

(vi) Let

$y = \sin^{-1}\left(\tan\frac{5\pi}{4}\right)$

Therefore above equation can be written as

$\text{Sin } y = \left(\tan\frac{5\pi}{4}\right) = \tan\left(\pi + \frac{\pi}{4}\right) = \tan\frac{\pi}{4} = 1 = \sin\left(\frac{\pi}{2}\right)$

We know that the principal value of $\sin^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\sin\left(\frac{\pi}{2}\right) = \tan\frac{5\pi}{4}$

Therefore the principal value of $\sin^{-1}\left(\tan\frac{5\pi}{4}\right)$ is $\frac{\pi}{2}$.

2.

(i) $\sin^{-1}\frac{1}{2} - 2\sin^{-1}\frac{1}{\sqrt{2}}$

(ii) $\sin^{-1}\left\{\cos\left(\sin^{-1}\frac{\sqrt{3}}{2}\right)\right\}$

Solution:

(i) The given question can be written as,

$$\sin^{-1}\frac{1}{2} - 2\sin^{-1}\frac{1}{\sqrt{2}} = \sin^{-1}\frac{1}{2} - \sin^{-1}\left(2 \times \frac{1}{\sqrt{2}}\sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2}\right)$$

On simplifying, we get

$$= \sin^{-1}\frac{1}{2} - \sin^{-1}(1)$$

By substituting the corresponding values, we get

$$= \frac{\pi}{6} - \frac{\pi}{2}$$

$$= -\frac{\pi}{3}$$

(ii) Given question can be written as

We know that $\left(\sin^{-1}\frac{\sqrt{3}}{2}\right) = \cos(\pi/3)$

$$= \sin^{-1}\left\{\cos\left(\frac{\pi}{3}\right)\right\}$$

Now substituting the values we get,

$$= \sin^{-1}\left\{\frac{\sqrt{3}}{2}\right\}$$

$$= \frac{\pi}{6}$$

EXERCISE 4.2 PAGE NO: 4.10

1. Find the domain of definition of $f(x) = \cos^{-1}(x^2 - 4)$

Solution:

Given $f(x) = \cos^{-1}(x^2 - 4)$
We know that domain of $\cos^{-1}(x^2 - 4)$ lies in the interval [-1, 1]
Therefore, we can write as
$-1 \leq x^2 - 4 \leq 1$
$4 - 1 \leq x^2 \leq 1 + 4$
$3 \leq x^2 \leq 5$
$\pm\sqrt{3} \leq x \leq \pm\sqrt{5}$
$-\sqrt{5} \leq x \leq -\sqrt{3}$ and $\sqrt{3} \leq x \leq \sqrt{5}$
Therefore domain of $\cos^{-1}(x^2 - 4)$ is $[-\sqrt{5}, -\sqrt{3}] \cup [\sqrt{3}, \sqrt{5}]$

2. Find the domain of $f(x) = \cos^{-1} 2x + \sin^{-1} x$.

Solution:

Given that $f(x) = \cos^{-1} 2x + \sin^{-1} x$.
Now we have to find the domain of f(x),
We know that domain of $\cos^{-1} x$ lies in the interval [-1, 1]
Also know that domain of $\sin^{-1} x$ lies I the interval [-1, 1]
Therefore, the domain of $\cos^{-1}(2x)$ lies in the interval [-1, 1]
Hence we can write as,
$-1 \leq 2x \leq 1$
$-\frac{1}{2} \leq x \leq \frac{1}{2}$
Hence domain $\cos^{-1}(2x) + \sin^{-1} x$ lies in the interval $[-\frac{1}{2}, \frac{1}{2}]$

EXERCISE 4.3 PAGE NO: 4.14

1. Find the principal value of each of the following:
(i) $\tan^{-1}$ (1/√3)
(ii) $\tan^{-1}$ (-1/√3)
(iii) $\tan^{-1}$ (cos (π/2))
(iv) $\tan^{-1}$ (2 cos (2π/3))

Solution:
(i) Given $\tan^{-1}$ (1/√3)
We know that for any x ∈ R, $\tan^{-1}$ represents an angle in (-π/2, π/2) whose tangent is x.
So, $\tan^{-1}$ (1/√3) = an angle in (-π/2, π/2) whose tangent is (1/√3)
But we know that the value is equal to π/6
Therefore $\tan^{-1}$ (1/√3) = π/6
Hence the principal value of $\tan^{-1}$ (1/√3) = π/6

(ii) Given $\tan^{-1}$ (-1/√3)
We know that for any x ∈ R, $\tan^{-1}$ represents an angle in (-π/2, π/2) whose tangent is x.
So, $\tan^{-1}$ (1-/√3) = an angle in (-π/2, π/2) whose tangent is (1/√3)
But we know that the value is equal to -π/6
Therefore $\tan^{-1}$ (-1/√3) = -π/6
Hence the principal value of $\tan^{-1}$ (-1/√3) = - π/6

(iii) Given that $\tan^{-1}$ (cos (π/2))
But we know that cos (π/2) = 0
We know that for any x ∈ R, $\tan^{-1}$ represents an angle in (-π/2, π/2) whose tangent is x.
Therefore $\tan^{-1}$ (0) = 0
Hence the principal value of $\tan^{-1}$ (cos (π/2) is 0.

(iv) Given that $\tan^{-1}$ (2 cos (2π/3))
But we know that cos π/3 = -1
Therefore $\tan^{-1}$ (2 cos (2π/3)) = $\tan^{-1}$ (2 × - ½)
= $\tan^{-1}$(-1)
= - π/4
Hence the principal value of $\tan^{-1}$ (2 cos (2π/3)) is - π/4

EXERCISE 4.4 PAGE NO: 4.18

1. Find the principal value of each of the following:
(i) $\sec^{-1}(-\sqrt{2})$
(ii) $\sec^{-1}(2)$
(iii) $\sec^{-1}(2\sin(3\pi/4))$
(iv) $\sec^{-1}(2\tan(3\pi/4))$

Solution:
(i) Given $\sec^{-1}(-\sqrt{2})$
Now let $y = \sec^{-1}(-\sqrt{2})$
$\sec y = -\sqrt{2}$
We know that $\sec \pi/4 = \sqrt{2}$
Therefore $-\sec(\pi/4) = \sqrt{2}$
$= \sec(\pi - \pi/4)$
$= \sec(3\pi/4)$
Thus the range of principal value of $\sec^{-1}$ is $[0, \pi] - \{\pi/2\}$
And $\sec(3\pi/4) = -\sqrt{2}$
Hence the principal value of $\sec^{-1}(-\sqrt{2})$ is $3\pi/4$

(ii) Given $\sec^{-1}(2)$
Let $y = \sec^{-1}(2)$
$\sec y = 2$
$= \sec \pi/3$
Therefore the range of principal value of $\sec^{-1}$ is $[0, \pi] - \{\pi/2\}$ and $\sec \pi/3 = 2$
Thus the principal value of $\sec^{-1}(2)$ is $\pi/3$

(iii) Given $\sec^{-1}(2\sin(3\pi/4))$
But we know that $\sin(3\pi/4) = 1/\sqrt{2}$
Therefore $2\sin(3\pi/4) = 2 \times 1/\sqrt{2}$
$2\sin(3\pi/4) = \sqrt{2}$
Therefore by substituting above values in $\sec^{-1}(2\sin(3\pi/4))$, we get
$\sec^{-1}(\sqrt{2})$
Let $\sec^{-1}(\sqrt{2}) = y$
$\sec y = \sqrt{2}$
$\sec(\pi/4) = \sqrt{2}$

Therefore range of principal value of $\sec^{-1}$ is $[0, \pi] - \{\pi/2\}$ and $\sec(\pi/4) = \sqrt{2}$
Thus the principal value of $\sec^{-1}(2\sin(3\pi/4))$ is $\pi/4$.

(iv) Given $\sec^{-1}(2\tan(3\pi/4))$
But we know that $\tan(3\pi/4) = -1$
Therefore, $2\tan(3\pi/4) = 2 \times -1$
$2\tan(3\pi/4) = -2$
By substituting these values in $\sec^{-1}(2\tan(3\pi/4))$, we get
$\text{Sec}^{-1}(-2)$
Now let $y = \text{Sec}^{-1}(-2)$
$\text{Sec } y = -2$
$-\sec(\pi/3) = 2$
$= \sec(\pi - \pi/3)$
$= \sec(2\pi/3)$
Therefore the range of principal value of $\sec^{-1}$ is $[0, \pi] - \{\pi/2\}$ and $\sec(2\pi/3) = -2$
Thus the principal value of $\sec^{-1}(2\tan(3\pi/4))$ is $(2\pi/3)$

EXERCISE 4.5 PAGE NO: 4.21

1. Find the principal values of each of the following:
(i) $\text{cosec}^{-1}(-\sqrt{2})$
(ii) $\text{cosec}^{-1}(-2)$
(iii) $\text{cosec}^{-1}(2/\sqrt{3})$
(iv) $\text{cosec}^{-1}(2\cos(2\pi/3))$

Solution:
(i) Given $\text{cosec}^{-1}(-\sqrt{2})$
Let $y = \text{cosec}^{-1}(-\sqrt{2})$
$\text{Cosec } y = -\sqrt{2}$
$-\text{Cosec } y = \sqrt{2}$
$-\text{Cosec}(\pi/4) = \sqrt{2}$
$-\text{Cosec}(\pi/4) = \text{cosec}(-\pi/4)$ [since $-\text{cosec }\theta = \text{cosec}(-\theta)$]
The range of principal value of cosec^{-1} $[-\pi/2, \pi/2] - \{0\}$ and $\text{cosec}(-\pi/4) = -\sqrt{2}$
$\text{Cosec}(-\pi/4) = -\sqrt{2}$
Therefore the principal value of $\text{cosec}^{-1}(-\sqrt{2})$ is $-\pi/4$

(ii) Given $\text{cosec}^{-1}(-2)$
Let $y = \text{cosec}^{-1}(-2)$
$\text{Cosec } y = -2$
$-\text{Cosec } y = 2$
$-\text{Cosec}(\pi/6) = 2$
$-\text{Cosec}(\pi/6) = \text{cosec}(-\pi/6)$ [since $-\text{cosec }\theta = \text{cosec}(-\theta)$]
The range of principal value of cosec^{-1} $[-\pi/2, \pi/2] - \{0\}$ and $\text{cosec}(-\pi/6) = -2$
$\text{Cosec}(-\pi/6) = -2$
Therefore the principal value of $\text{cosec}^{-1}(-2)$ is $-\pi/6$

(iii) Given $\text{cosec}^{-1}(2/\sqrt{3})$
Let $y = \text{cosec}^{-1}(2/\sqrt{3})$
$\text{Cosec } y = (2/\sqrt{3})$
$\text{Cosec}(\pi/3) = (2/\sqrt{3})$
Therefore range of principal value of cosec^{-1} is $[-\pi/2, \pi/2] - \{0\}$ and $\text{cosec}(\pi/3) = (2/\sqrt{3})$
Thus, the principal value of $\text{cosec}^{-1}(2/\sqrt{3})$ is $\pi/3$

(iv) Given $cosec^{-1}$ (2 cos (2π/3))
But we know that cos (2π/3) = - ½
Therefore 2 cos (2π/3) = 2 × - ½
2 cos (2π/3) = -1
By substituting these values in $cosec^{-1}$ (2 cos (2π/3)) we get,
$Cosec^{-1}$ (-1)
Let y = $cosec^{-1}$ (-1)
- Cosec y = 1
- Cosec (π/2) = cosec (-π/2) [since –cosec θ = cosec (-θ)]
The range of principal value of $cosec^{-1}$ [-π/2, π/2] – {0} and cosec (-π/2) = - 1
Cosec (-π/2) = - 1
Therefore the principal value of $cosec^{-1}$ (2 cos (2π/3)) is - π/2

EXERCISE 4.6 PAGE NO: 4.24

1. Find the principal values of each of the following:
(i) $\cot^{-1}(-\sqrt{3})$
(ii) $\text{Cot}^{-1}(\sqrt{3})$
(iii) $\cot^{-1}(-1/\sqrt{3})$
(iv) $\cot^{-1}(\tan 3\pi/4)$

Solution:
(i) Given $\cot^{-1}(-\sqrt{3})$
Let $y = \cot^{-1}(-\sqrt{3})$
$-\text{Cot}(\pi/6) = \sqrt{3}$
$= \text{Cot}(\pi - \pi/6)$
$= \cot(5\pi/6)$
The range of principal value of $\cot^{-1}$ is $(0, \pi)$ and $\cot(5\pi/6) = -\sqrt{3}$
Thus, the principal value of $\cot^{-1}(-\sqrt{3})$ is $5\pi/6$

(ii) Given $\text{Cot}^{-1}(\sqrt{3})$
Let $y = \cot^{-1}(\sqrt{3})$
$\text{Cot}(\pi/6) = \sqrt{3}$
The range of principal value of $\cot^{-1}$ is $(0, \pi)$ and
Thus, the principal value of $\cot^{-1}(\sqrt{3})$ is $\pi/6$

(iii) Given $\cot^{-1}(-1/\sqrt{3})$
Let $y = \cot^{-1}(-1/\sqrt{3})$
$\text{Cot } y = (-1/\sqrt{3})$
$-\text{Cot}(\pi/3) = 1/\sqrt{3}$
$= \text{Cot}(\pi - \pi/3)$
$= \cot(2\pi/3)$
The range of principal value of $\cot^{-1}(0, \pi)$ and $\cot(2\pi/3) = -1/\sqrt{3}$
Therefore the principal value of $\cot^{-1}(-1/\sqrt{3})$ is $2\pi/3$

(iv) Given $\cot^{-1}(\tan 3\pi/4)$
But we know that $\tan 3\pi/4 = -1$
By substituting this value in $\cot^{-1}(\tan 3\pi/4)$ we get
$\text{Cot}^{-1}(-1)$

Now, let y = $\cot^{-1}(-1)$
Cot y = (-1)
- Cot (π/4) = 1
= Cot (π – π/4)
= cot (3π/4)
The range of principal value of $\cot^{-1}(0, \pi)$ and cot (3π/4) = - 1
Therefore the principal value of $\cot^{-1}(\tan 3\pi/4)$ is 3π/4

EXERCISE 4.7

1. Evaluate each of the following:
(i) $\sin^{-1}(\sin \pi/6)$
(ii) $\sin^{-1}(\sin 7\pi/6)$
(iii) $\sin^{-1}(\sin 5\pi/6)$
(iv) $\sin^{-1}(\sin 13\pi/7)$
(v) $\sin^{-1}(\sin 17\pi/8)$
(vi) $\sin^{-1}\{(\sin - 17\pi/8)\}$
(vii) $\sin^{-1}(\sin 3)$
(viii) $\sin^{-1}(\sin 4)$
(ix) $\sin^{-1}(\sin 12)$
(x) $\sin^{-1}(\sin 2)$

Solution:
(i) Given $\sin^{-1}(\sin \pi/6)$
We know that the value of sin π/6 is ½
By substituting this value in $\sin^{-1}(\sin \pi/6)$
We get, $\sin^{-1}(1/2)$
Now let $y = \sin^{-1}(1/2)$
Sin (π/6) = ½
The range of principal value of $\sin^{-1}(-\pi/2, \pi/2)$ and sin (π/6) = ½
Therefore $\sin^{-1}(\sin \pi/6) = \pi/6$

(ii) Given $\sin^{-1}(\sin 7\pi/6)$
But we know that sin 7π/6 = - ½
By substituting this in $\sin^{-1}(\sin 7\pi/6)$ we get,
$\text{Sin}^{-1}(-1/2)$
Now let $y = \sin^{-1}(-1/2)$
- Sin y = ½
- Sin (π/6) = ½
- Sin (π/6) = sin (- π/6)
The range of principal value of $\sin^{-1}(-\pi/2, \pi/2)$ and sin (- π/6) = - ½
Therefore $\sin^{-1}(\sin 7\pi/6) = -\pi/6$

(iii) Given $\sin^{-1}(\sin 5\pi/6)$

We know that the value of sin 5π/6 is ½
By substituting this value in $\sin^{-1}$(sin π/6)
We get, $\sin^{-1}$ (1/2)
Now let y = $\sin^{-1}$ (1/2)
Sin (π/6) = ½
The range of principal value of $\sin^{-1}$(-π/2, π/2) and sin (π/6) = ½
Therefore $\sin^{-1}$(sin 5π/6) = π/6

(iv) Given $\sin^{-1}$(sin 13π/7)
Given question can be written as sin (2π – π/7)
Sin (2π – π/7) can be written as sin (π/7) [since sin (2π – θ) = sin (-θ)]
By substituting these values in $\sin^{-1}$(sin 13π/7) we get $\sin^{-1}$(sin - π/7)
As $\sin^{-1}$(sin x) = x with x ∈ [-π/2, π/2]
Therefore $\sin^{-1}$(sin 13π/7) = - π/7

(v) Given $\sin^{-1}$(sin 17π/8)
Given question can be written as sin (2π + π/8)
Sin (2π + π/8) can be written as sin (π/8)
By substituting these values in $\sin^{-1}$(sin 17π/8) we get $\sin^{-1}$(sin π/8)
As $\sin^{-1}$(sin x) = x with x ∈ [-π/2, π/2]
Therefore $\sin^{-1}$(sin 17π/8) = π/8

(vi) Given $\sin^{-1}${(sin - 17π/8)}
But we know that – sin θ = sin (-θ)
Therefore (sin -17π/8) = - sin 17π/8
- Sin 17π/8 = - sin (2π + π/8) [since sin (2π – θ) = sin (θ)]
It can also be written as – sin (π/8)
– Sin (π/8) = sin (-π/8) [since – sin θ = sin (-θ)]
By substituting these values in $\sin^{-1}${(sin - 17π/8)} we get,
Sin^{-1}(sin - π/8)
As $\sin^{-1}$(sin x) = x with x ∈ [-π/2, π/2]
Therefore $\sin^{-1}$(sin -π/8) = - π/8

(vii) Given $\sin^{-1}$(sin 3)
We know that $\sin^{-1}$(sin x) = x with x ∈ [-π/2, π/2] which is approximately equal to [-1.57, 1.57]
But here x = 3, which does not lie on the above range,

Therefore we know that sin (π – x) = sin (x)
Hence sin (π – 3) = sin (3) also π – 3 ∈ [-π/2, π/2]
Sin^{-1}(sin 3) = π – 3

(viii) Given sin^{-1}(sin 4)
We know that sin^{-1}(sin x) = x with x ∈ [-π/2, π/2] which is approximately equal to [-1.57, 1.57]
But here x = 4, which does not lie on the above range,
Therefore we know that sin (π – x) = sin (x)
Hence sin (π – 4) = sin (4) also π – 4 ∈ [-π/2, π/2]
Sin^{-1}(sin 4) = π – 4

(ix) Given sin^{-1}(sin 12)
We know that sin^{-1}(sin x) = x with x ∈ [-π/2, π/2] which is approximately equal to [-1.57, 1.57]
But here x = 12, which does not lie on the above range,
Therefore we know that sin (2nπ – x) = sin (-x)
Hence sin (2nπ – 12) = sin (-12)
Here n = 2 also 12 – 4π ∈ [-π/2, π/2]
Sin^{-1}(sin 12) = 12 – 4π

(x) Given sin^{-1}(sin 2)
We know that sin^{-1}(sin x) = x with x ∈ [-π/2, π/2] which is approximately equal to [-1.57, 1.57]
But here x = 2, which does not lie on the above range,
Therefore we know that sin (π – x) = sin (x)
Hence sin (π – 2) = sin (2) also π – 2 ∈ [-π/2, π/2]
Sin^{-1}(sin 2) = π – 2

2. Evaluate each of the following:
(i) cos^{-1}{cos (-π/4)}
(ii) cos^{-1}(cos 5π/4)
(iii) cos^{-1}(cos 4π/3)
(iv) cos^{-1}(cos 13π/6)
(v) cos^{-1}(cos 3)
(vi) cos^{-1}(cos 4)
(vii) cos^{-1}(cos 5)

(viii) $\cos^{-1}(\cos 12)$

Solution:
(i) Given $\cos^{-1}\{\cos(-\pi/4)\}$
We know that $\cos(-\pi/4) = \cos(\pi/4)$ [since $\cos(-\theta) = \cos\theta$
Also know that $\cos(\pi/4) = 1/\sqrt{2}$
By substituting these values in $\cos^{-1}\{\cos(-\pi/4)\}$ we get,
$\text{Cos}^{-1}(1/\sqrt{2})$
Now let $y = \cos^{-1}(1/\sqrt{2})$
Therefore $\cos y = 1/\sqrt{2}$
Hence range of principal value of $\cos^{-1}$ is $[0, \pi]$ and $\cos(\pi/4) = 1/\sqrt{2}$
Therefore $\cos^{-1}\{\cos(-\pi/4)\} = \pi/4$

(ii) Given $\cos^{-1}(\cos 5\pi/4)$
But we know that $\cos(5\pi/4) = -1/\sqrt{2}$
By substituting these values in $\cos^{-1}\{\cos(5\pi/4)\}$ we get,
$\text{Cos}^{-1}(-1/\sqrt{2})$
Now let $y = \cos^{-1}(-1/\sqrt{2})$
Therefore $\cos y = -1/\sqrt{2}$
$-\text{Cos}(\pi/4) = 1/\sqrt{2}$
$\text{Cos}(\pi - \pi/4) = -1/\sqrt{2}$
$\text{Cos}(3\pi/4) = -1/\sqrt{2}$
Hence range of principal value of $\cos^{-1}$ is $[0, \pi]$ and $\cos(3\pi/4) = -1/\sqrt{2}$
Therefore $\cos^{-1}\{\cos(5\pi/4)\} = 3\pi/4$

(iii) Given $\cos^{-1}(\cos 4\pi/3)$
But we know that $\cos(4\pi/3) = -1/2$
By substituting these values in $\cos^{-1}\{\cos(4\pi/3)\}$ we get,
$\text{Cos}^{-1}(-1/2)$
Now let $y = \cos^{-1}(-1/2)$
Therefore $\cos y = -1/2$
$-\text{Cos}(\pi/3) = 1/2$
$\text{Cos}(\pi - \pi/3) = -1/2$
$\text{Cos}(2\pi/3) = -1/2$
Hence range of principal value of $\cos^{-1}$ is $[0, \pi]$ and $\cos(2\pi/3) = -1/2$
Therefore $\cos^{-1}\{\cos(4\pi/3)\} = 2\pi/3$

(iv) Given $\cos^{-1}(\cos 13\pi/6)$
But we know that $\cos (13\pi/6) = \sqrt{3}/2$
By substituting these values in $\cos^{-1}\{\cos (13\pi/6)\}$ we get,
$\text{Cos}^{-1}(\sqrt{3}/2)$
Now let $y = \cos^{-1}(\sqrt{3}/2)$
Therefore $\cos y = \sqrt{3}/2$
$\text{Cos}(\pi/6) = \sqrt{3}/2$
Hence range of principal value of $\cos^{-1}$ is $[0, \pi]$ and $\cos (\pi/6) = \sqrt{3}/2$
Therefore $\cos^{-1}\{\cos (13\pi/6)\} = \pi/6$

(v) Given $\cos^{-1}(\cos 3)$
We know that $\cos^{-1}(\cos \theta) = \theta$ if $0 \le \theta \le \pi$
Therefore by applying this in given question we get,
$\text{Cos}^{-1}(\cos 3) = 3, 3 \in [0, \pi]$

(vi) Given $\cos^{-1}(\cos 4)$
We have $\cos^{-1}(\cos x) = x$ if $x \in [0, \pi] \approx [0, 3.14]$
And here $x = 4$ which does not lie in the above range.
We know that $\cos (2\pi - x) = \cos(x)$
Thus, $\cos (2\pi - 4) = \cos (4)$ so $2\pi - 4$ belongs in $[0, \pi]$
Hence $\cos^{-1}(\cos 4) = 2\pi - 4$

(vii) Given $\cos^{-1}(\cos 5)$
We have $\cos^{-1}(\cos x) = x$ if $x \in [0, \pi] \approx [0, 3.14]$
And here $x = 5$ which does not lie in the above range.
We know that $\cos (2\pi - x) = \cos(x)$
Thus, $\cos (2\pi - 5) = \cos (5)$ so $2\pi - 5$ belongs in $[0, \pi]$
Hence $\cos^{-1}(\cos 5) = 2\pi - 5$

(viii) Given $\cos^{-1}(\cos 12)$
$\text{Cos}^{-1}(\cos x) = x$ if $x \in [0, \pi] \approx [0, 3.14]$
And here $x = 12$ which does not lie in the above range.
We know $\cos (2n\pi - x) = \cos (x)$
$\text{Cos}(2n\pi - 12) = \cos (12)$
Here $n = 2$.
Also $4\pi - 12$ belongs in $[0, \pi]$
$\therefore \cos^{-1}(\cos 12) = 4\pi - 12$

3. Evaluate each of the following:
(i) $\tan^{-1}(\tan \pi/3)$
(ii) $\tan^{-1}(\tan 6\pi/7)$
(iii) $\tan^{-1}(\tan 7\pi/6)$
(iv) $\tan^{-1}(\tan 9\pi/4)$
(v) $\tan^{-1}(\tan 1)$
(vi) $\tan^{-1}(\tan 2)$
(vii) $\tan^{-1}(\tan 4)$
(viii) $\tan^{-1}(\tan 12)$

Solution:
(i) Given $\tan^{-1}(\tan \pi/3)$
As $\tan^{-1}(\tan x) = x$ if $x \in [-\pi/2, \pi/2]$
By applying this condition in the given question we get,
$\text{Tan}^{-1}(\tan \pi/3) = \pi/3$

(ii) Given $\tan^{-1}(\tan 6\pi/7)$
We know that tan $6\pi/7$ can be written as $(\pi - \pi/7)$
$\text{Tan}(\pi - \pi/7) = -\tan \pi/7$
We know that $\tan^{-1}(\tan x) = x$ if $x \in [-\pi/2, \pi/2]$
$\text{Tan}^{-1}(\tan 6\pi/7) = -\pi/7$

(iii) Given $\tan^{-1}(\tan 7\pi/6)$
We know that $\tan 7\pi/6 = 1/\sqrt{3}$
By substituting this value in $\tan^{-1}(\tan 7\pi/6)$ we get,
$\text{Tan}^{-1}(1/\sqrt{3})$
Now let $\tan^{-1}(1/\sqrt{3}) = y$
$\text{Tan } y = 1/\sqrt{3}$
$\text{Tan}(\pi/6) = 1/\sqrt{3}$
The range of the principal value of $\tan^{-1}$ is $(-\pi/2, \pi/2)$ and $\tan(\pi/6) = 1/\sqrt{3}$
Therefore $\tan^{-1}(\tan 7\pi/6) = \pi/6$

(iv) Given $\tan^{-1}(\tan 9\pi/4)$
We know that $\tan 9\pi/4 = 1$
By substituting this value in $\tan^{-1}(\tan 9\pi/4)$ we get,
$\text{Tan}^{-1}(1)$
Now let $\tan^{-1}(1) = y$

Tan y = 1
Tan (π/4) = 1
The range of the principal value of $\tan^{-1}$ is (-π/2, π/2) and tan (π/4) = 1
Therefore $\tan^{-1}$(tan 9π/4) = π/4

(v) Given $\tan^{-1}$(tan 1)
But we have $\tan^{-1}$(tan x) = x if x ϵ [-π/2, π/2]
By substituting this condition in given question
Tan^{-1}(tan 1) = 1

(vi) Given $\tan^{-1}$(tan 2)
As $\tan^{-1}$(tan x) = x if x ϵ [-π/2, π/2]
But here x = 2 which does not belongs to above range
We also have tan (π – θ) = –tan (θ)
Therefore tan (θ – π) = tan (θ)
Tan (2 – π) = tan (2)
Now 2 – π is in the given range
Hence $\tan^{-1}$ (tan 2) = 2 – π

(vii) Given $\tan^{-1}$(tan 4)
As $\tan^{-1}$(tan x) = x if x ϵ [-π/2, π/2]
But here x = 4 which does not belongs to above range
We also have tan (π – θ) = –tan (θ)
Therefore tan (θ – π) = tan (θ)
Tan (4 – π) = tan (4)
Now 4 – π is in the given range
Hence $\tan^{-1}$ (tan 2) = 4 – π

(viii) Given $\tan^{-1}$(tan 12)
As $\tan^{-1}$(tan x) = x if x ϵ [-π/2, π/2]
But here x = 12 which does not belongs to above range
We know that tan (nπ – θ) = –tan (θ)
Tan (θ – 2nπ) = tan (θ)
Here n = 4
Tan (12 – 4π) = tan (12)
Now 12 – 4π is in the given range
∴ $\tan^{-1}$ (tan 12) = 12 – 4π.

EXERCISE 4.8 PAGE NO: 4.54

1. Evaluate each of the following:
(i) sin ($\sin^{-1}$ 7/25)
(ii) Sin ($\cos^{-1}$ 5/13)
(iii) Sin ($\tan^{-1}$ 24/7)
(iv) Sin ($\sec^{-1}$ 17/8)
(v) Cosec ($\cos^{-1}$ 8/17)
(vi) Sec ($\sin^{-1}$ 12/13)
(vii) Tan ($\cos^{-1}$ 8/17)
(viii) cot ($\cos^{-1}$ 3/5)
(ix) Cos ($\tan^{-1}$ 24/7)

Solution:
(i) Given sin ($\sin^{-1}$ 7/25)
Now let y = $\sin^{-1}$ 7/25
Sin y = 7/25 where y ∈ [0, π/2]
Substituting these values in sin ($\sin^{-1}$ 7/25) we get
Sin ($\sin^{-1}$ 7/25) = 7/25

(ii) Given Sin ($\cos^{-1}$ 5/13)

Let $\cos^{-1}\frac{5}{13} = y$

$\Rightarrow \cos y = \frac{5}{13}$ Where $y \in \left[0, \frac{\pi}{2}\right]$

Now we have to find

$$\sin\left(\cos^{-1}\frac{5}{13}\right) = \sin y$$

We know that $\sin^2\theta + \cos^2\theta = 1$

By substituting this trigonometric identity we get

$\Rightarrow \sin y = \pm\sqrt{1-\cos^2 y}$

Where $y \in \left[0, \frac{\pi}{2}\right]$

$\Rightarrow \sin y = \sqrt{1-\cos^2 y}$

Now by substituting cos y value we get

$\Rightarrow \sin y = \sqrt{1-\left(\frac{5}{13}\right)^2}$

$\Rightarrow \sin y = \sqrt{1-\frac{25}{169}}$

$\Rightarrow \sin y = \sqrt{\frac{144}{169}}$

$\Rightarrow \sin y = \frac{12}{13} \Rightarrow \sin\left[\cos^{-1}\left(\frac{5}{13}\right)\right] = \frac{12}{13}$

(iii) Given Sin ($\tan^{-1}$ 24/7)

Let $\tan^{-1}\frac{24}{7} = y$

$\Rightarrow \tan y = \frac{24}{7}$ Where $y \in \left[0, \frac{\pi}{2}\right]$

Now we have to find

$$\sin\left(\tan^{-1}\frac{24}{7}\right) = \sin y$$

We know that 1 + $\cot^2\theta = \text{cosec}^2\theta$

⇒ 1 + $\cot^2 y = \text{cosec}^2 y$

Now substituting this trigonometric identity we get,

$$\Rightarrow 1+\left(\frac{7}{24}\right)^2 = \text{cosec}^2 y$$

$$\Rightarrow 1+\frac{49}{576} = \frac{1}{\sin^2 y}$$

On rearranging we get,

$$\Rightarrow \sin^2 y = \frac{576}{625}$$

$$\Rightarrow \sin y = \frac{24}{25} \text{ Where } y \in \left[0, \frac{\pi}{2}\right]$$

$$\Rightarrow \sin\left(\tan^{-1}\frac{24}{7}\right) = \frac{24}{25}$$

(iv) Given Sin ($\sec^{-1}$ 17/8)

Let $\sec^{-1}\frac{17}{8} = y$

$$\Rightarrow \sec y = \frac{17}{8} \text{ Where } y \in \left[0, \frac{\pi}{2}\right]$$

Now we have find

$\sin\left(\sec^{-1}\frac{17}{8}\right) = \sin y$

We know that, $\cos y = \frac{1}{\sec y}$

$\Rightarrow \cos y = \frac{8}{17}$

Now, $\sin y = \sqrt{1-\cos^2 y}$ where $y \in \left[0, \frac{\pi}{2}\right]$

By substituting, cos y value we get,

$\Rightarrow \sin y = \sqrt{1-\left(\frac{8}{17}\right)^2}$

$\Rightarrow \sin y = \sqrt{\frac{225}{289}}$

$\Rightarrow \sin y = \frac{15}{17}$

$\Rightarrow \sin\left(\sec^{-1}\frac{17}{8}\right) = \frac{15}{17}$

(v) Given Cosec ($\cos^{-1}$ 8/17)

Let $\cos^{-1}\frac{3}{5} = y$

$\Rightarrow \cos y = \frac{3}{5}$ Where $y \in \left[0, \frac{\pi}{2}\right]$

Now we have to find

$\operatorname{cosec}\left(\cos^{-1}\frac{3}{5}\right) = \operatorname{cosec} y$

We know that $\sin^2\theta + \cos^2\theta = 1$

On rearranging and substituting we get,

$\Rightarrow \sin y = \sqrt{1-\cos^2 y}$ Where $y \in \left[0, \frac{\pi}{2}\right]$

Now substituting the value of cos y we get

$$\Rightarrow \sin y = \sqrt{1-\left(\frac{3}{5}\right)^2}$$

$$\Rightarrow \sin y = \sqrt{\frac{16}{25}}$$

$$\Rightarrow \sin y = \frac{4}{5}$$

$$\Rightarrow \operatorname{cosec} y = \frac{5}{4}$$

$$\Rightarrow \operatorname{cosec}\left(\cos^{-1}\frac{3}{5}\right) = \frac{5}{4}$$

(vi) Given Sec ($\sin^{-1}$ 12/13)

Let $\sin^{-1}\frac{12}{13} = y$ where $y \in \left[0, \frac{\pi}{2}\right]$

$$\Rightarrow \sin y = \frac{12}{13}$$

Now we have to find

$$\sec\left(\sin^{-1}\frac{12}{13}\right) = \sec y$$

We know that $\sin^2\theta + \cos^2\theta = 1$

According to this identity cos y can be written as

$\Rightarrow \cos y = \sqrt{1-\sin^2 y}$ Where $y \in \left[0, \frac{\pi}{2}\right]$

Now substituting the value of sin y we get,

$\Rightarrow \cos y = \sqrt{1-\left(\frac{12}{13}\right)^2}$

$\Rightarrow \cos y = \sqrt{1-\frac{144}{169}}$

$\Rightarrow \cos y = \sqrt{\frac{25}{169}}$

$\Rightarrow \cos y = \frac{5}{13}$

$\Rightarrow \sec y = \frac{1}{\cos y}$

$\Rightarrow \sec y = \frac{13}{5}$

$\Rightarrow \sec\left(\sin^{-1}\frac{12}{13}\right) = \frac{13}{5}$

(vii) Given Tan ($\cos^{-1}$ 8/17)

Let $\cos^{-1}\frac{8}{17} = y$ where $y \in \left[0, \frac{\pi}{2}\right]$

$\Rightarrow \cos y = \frac{8}{17}$

Now we have to find

$\tan\left(\cos^{-1}\frac{8}{17}\right) = \tan y$

We know that $1+\tan^2\theta = \sec^2\theta$

Rearranging and substituting the value of tan y we get,

$\Rightarrow \tan y = \sqrt{\sec^2 y - 1}$ Where $y \in \left[0, \frac{\pi}{2}\right]$

We have sec y = 1/cos y

$\Rightarrow \tan y = \sqrt{\left(\frac{1}{\cos^2 y}\right) - 1}$

$\Rightarrow \tan y = \sqrt{\left(\frac{17}{8}\right)^2 - 1}$

$\Rightarrow \tan y = \sqrt{\frac{289}{64} - 1}$

$\Rightarrow \tan y = \sqrt{\frac{225}{64}}$

$\Rightarrow \tan y = \frac{15}{8}$

$\Rightarrow \tan\left(\cos^{-1}\frac{8}{17}\right) = \frac{15}{8}$

(viii) Given cot ($\cos^{-1}$ 3/5)

Let $\cos^{-1}\frac{3}{5} = y$ where $y \in \left[0, \frac{\pi}{2}\right]$

$\Rightarrow \cos y = \frac{3}{5}$

Now we have to find

$\cot\left(\cos^{-1}\frac{3}{5}\right) = \cot y$

We know that $1+\tan^2\theta = \sec^2\theta$

Rearranging and substituting the value of tan y we get,

$\Rightarrow \tan y = \sqrt{\sec^2 y - 1}$ Where $y \in \left[0, \frac{\pi}{2}\right]$

We have sec y = 1/cos y, on substitution we get,

$\Rightarrow \frac{1}{\cot y} = \sqrt{\left(\frac{1}{\cos^2 y}\right) - 1}$

$\Rightarrow \frac{1}{\cot y} = \sqrt{\left(\frac{5}{3}\right)^2 - 1}$

$\Rightarrow \frac{1}{\cot y} = \sqrt{\frac{16}{9}}$

$\Rightarrow \cot y = \frac{3}{4}$

$\Rightarrow \cot\left(\cos^{-1}\frac{3}{5}\right) = \frac{3}{4}$

(ix) Given Cos ($\tan^{-1}$ 24/7)

Let $\tan^{-1}\frac{24}{7} = y$

$\Rightarrow \tan y = \frac{24}{7}$ Where $y \in \left[0, \frac{\pi}{2}\right]$

Now we have to find,

$\cos\left(\tan^{-1}\frac{24}{7}\right) = \cos y$

We know that $1+\tan^2\theta = \sec^2\theta$

$\Rightarrow 1+\tan^2 y = \sec^2 y$

On rearranging and substituting the value of sec y we get,

$\Rightarrow \sec y = \sqrt{1+\tan^2 y}$ Where $y \in \left[0, \frac{\pi}{2}\right]$

$\Rightarrow \sec y = \sqrt{1+\left(\frac{24}{7}\right)^2}$

$\Rightarrow \sec y = \sqrt{\frac{625}{49}}$

$\Rightarrow \sec y = \frac{25}{7}$

$\Rightarrow \cos y = \frac{1}{\sec y}$

$\Rightarrow \cos y = \frac{7}{25}$

$\Rightarrow \cos\left(\tan^{-1}\frac{24}{7}\right) = \frac{7}{25}$

EXERCISE 4.9 PAGE NO: 4.58

1. Evaluate:
(i) Cos {sin^{-1} (-7/25)}
(ii) Sec {cot^{-1} (-5/12)}
(iii) Cot {sec^{-1} (-13/5)}

Solution:
(i) Given Cos {sin^{-1} (-7/25)}

Let $\sin^{-1}\left(-\frac{7}{25}\right) = x$ Where $x \in \left[-\frac{\pi}{2}, 0\right]$

$\Rightarrow \sin x = -\frac{7}{25}$

Now we have to find

$$\cos\left[\sin^{-1}\left(-\frac{7}{25}\right)\right] = \cos x$$

We know that $\sin^2 x + \cos^2 x = 1$
On rearranging and substituting we get,

$\Rightarrow \cos x = \sqrt{1 - \sin^2 x}$ Since $x \in \left[-\frac{\pi}{2}, 0\right]$

$\Rightarrow \cos x = \sqrt{1 - \frac{49}{625}}$

$\Rightarrow \cos x = \sqrt{\frac{576}{625}}$

$\Rightarrow \cos x = \frac{24}{25}$

$\Rightarrow \cos\left[\sin^{-1}\left(-\frac{7}{25}\right)\right] = \frac{24}{25}$

(ii) Given Sec {cot^{-1} (-5/12)}

Let $\cot^{-1}\left(-\frac{5}{12}\right) = x$ where $x \in \left(\frac{\pi}{2}, \pi\right)$

$\Rightarrow \cot x = -\frac{5}{12}$

Now we have to find,

$\sec\left[\cot^{-1}\left(-\frac{5}{12}\right)\right] = \sec x$

We know that $1 + \tan^2 x = \sec^2 x$

On rearranging, we get

$\Rightarrow 1 + \frac{1}{\cot^2 x} = \sec^2 x$

Substituting these values we get,

$\Rightarrow \sec x = -\sqrt{1 + \frac{1}{\cot^2 x}}$ Since $x \in \left(\frac{\pi}{2}, \pi\right)$

$\Rightarrow \sec x = -\sqrt{1 + \left(\frac{12}{5}\right)^2}$

$\Rightarrow \sec x = -\frac{13}{5}$

$\Rightarrow \sec\left[\cot^{-1}\left(-\frac{5}{12}\right)\right] = -\frac{13}{5}$

(iii) Given Cot {sec^{-1} (-13/5)}

Let $\sec^{-1}\left(-\frac{13}{5}\right) = x$ where $x \in \left(\frac{\pi}{2}, \pi\right)$

$\Rightarrow \sec x = -\frac{13}{5}$

Now we have find,

$\cot\left[\sec^{-1}\left(-\frac{13}{5}\right)\right] = \cot x$

We know that $1 + \tan^2 x = \sec^2 x$

On rearranging, we get

$\Rightarrow \tan x = -\sqrt{\sec^2 x - 1}$

Now substitute the value of sec x, we get

$\Rightarrow \tan x = -\sqrt{\left(-\frac{13}{5}\right)^2 - 1}$

$\Rightarrow \tan x = -\frac{12}{5}$

$\Rightarrow \cot x = -\frac{5}{12}$

$\Rightarrow \cot\left[\sec^{-1}\left(-\frac{13}{5}\right)\right] = -\frac{5}{12}$

EXERCISE 4.10 PAGE NO: 4.66

1. Evaluate:
(i) Cot ($\sin^{-1}$ (3/4) + $\sec^{-1}$ (4/3))
(ii) Sin ($\tan^{-1}$ x + $\tan^{-1}$ 1/x) for x < 0
(iii) Sin ($\tan^{-1}$ x + $\tan^{-1}$ 1/x) for x > 0
(iv) Cot ($\tan^{-1}$ a + $\cot^{-1}$ a)
(v) Cos ($\sec^{-1}$ x + $\operatorname{cosec}^{-1}$ x), |x| ≥ 1

Solution:
(i) Given Cot ($\sin^{-1}$ (3/4) + $\sec^{-1}$ (4/3))

$$= \cot\left(\sin^{-1}\frac{3}{4}+\cos^{-1}\frac{3}{4}\right)$$

$$\left(\because \sec^{-1}x=\cos^{-1}\frac{1}{x}\right)$$

We have

$$\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$$

By substituting these values in given questions, we get

$$= \cot\frac{\pi}{2}$$

= 0

(ii) Given Sin ($\tan^{-1}$ x + $\tan^{-1}$ 1/x) for x < 0

$$= \sin\left(\tan^{-1}x+(\cot^{-1}x-\pi)\right)\left(\because \tan^{-1}\theta=\cot^{-1}\frac{1}{\theta}-\pi \qquad \text{for } x<0\right)$$

$= \sin\left(\frac{\pi}{2} - \pi\right)\left(\because \tan^{-1}\theta + \cot^{-1}\theta = \frac{\pi}{2}\right)$

On simplifying, we get

$= \sin\left(-\frac{\pi}{2}\right)$

We know that sin (- θ) = – sin θ

$= -\sin\frac{\pi}{2} = -1$

(iii) Given Sin ($\tan^{-1}$ x + $\tan^{-1}$ 1/x) for x > 0

$= \sin\left(\tan^{-1}x + \cot^{-1}x\right)\left(\because \tan^{-1}\theta = \cot^{-1}\frac{1}{\theta} \quad \text{for } x > 0\right)$

Again we know that,

$\tan^{-1}\theta + \cot^{-1}\theta = \frac{\pi}{2}$

Now by substituting above identity in given question we get,

$= \sin\frac{\pi}{2}$

= 1

(iv) Given Cot ($\tan^{-1}$ a + $\cot^{-1}$ a)

We know that,

$\tan^{-1}\theta + \cot^{-1}\theta = \frac{\pi}{2}$

Now by substituting above identity in given question we get,

$= \cot\left(\frac{\pi}{2}\right)$

= 0

(v) Given Cos ($\sec^{-1} x + \text{cosec}^{-1} x$), |x| ≥ 1

We know that

$\sec^{-1}\theta = \cos^{-1}\frac{1}{\theta}$

Again we have

$\text{cosec}^{-1}\theta = \sin^{-1}\frac{1}{\theta}$

By substituting these values in given question we get,

$= \cos\left(\cos^{-1}\frac{1}{x} + \sin^{-1}\frac{1}{x}\right)$

We know that from the identities,

$\sin^{-1}\theta + \cos^{-1}\theta = \frac{\pi}{2}$

Now by substituting we get,

$= \cos\frac{\pi}{2}$

2. If $\cos^{-1} x + \cos^{-1} y = \pi/4$, find the value of $\sin^{-1} x + \sin^{-1} y$.

Solution:

Given $\cos^{-1} x + \cos^{-1} y = \pi/4$

We know that

$$\sin^{-1}\theta + \cos^{-1}\theta = \frac{\pi}{2}$$

Now substituting above identity in given question we get,

$$\Rightarrow \left(\frac{\pi}{2} - \sin^{-1}x\right) + \left(\frac{\pi}{2} - \sin^{-1}y\right) = \frac{\pi}{4}$$

Adding and simplifying we get,

$$\Rightarrow \pi - \left(\sin^{-1}x + \sin^{-1}y\right) = \frac{\pi}{4}$$

On rearranging,

$$\Rightarrow \sin^{-1}x + \sin^{-1}y = \pi - \frac{\pi}{4}$$

$$\Rightarrow \sin^{-1}x + \sin^{-1}y = \frac{3\pi}{4}$$

3. If $\sin^{-1} x + \sin^{-1} y = \pi/3$ and $\cos^{-1} x - \cos^{-1} y = \pi/6$, find the values of x and y.

Solution:

Given $\sin^{-1} x + \sin^{-1} y = \pi/3$ ……. Equation (i)
And $\cos^{-1} x - \cos^{-1} y = \pi/6$ ……… Equation (ii)
Subtracting Equation (ii) from Equation (i), we get

$$\Rightarrow (\sin^{-1}x - \cos^{-1}x) + (\sin^{-1}y + \cos^{-1}y) = \frac{\pi}{3} - \frac{\pi}{6}$$

We know that,

$$\sin^{-1}\theta + \cos^{-1}\theta = \frac{\pi}{2}$$

By substituting above identity, we get

$$\Rightarrow (\sin^{-1} x - \cos^{-1} x) + \left(\frac{\pi}{2}\right) = \frac{\pi}{6}$$

Replacing $\sin^{-1} x$ by $\pi/2 - \cos^{-1} x$ and rearranging we get,

$$\Rightarrow \left(\frac{\pi}{2} - \cos^{-1} x\right) - \cos^{-1} x = -\frac{\pi}{3}$$

Now by adding,

$$\Rightarrow 2\cos^{-1} x = \frac{5\pi}{6}$$

$$\Rightarrow \cos^{-1} x = \frac{5\pi}{12}$$

$$\Rightarrow x = \cos\left(\frac{5\pi}{12}\right)$$

$$\Rightarrow x = \cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right)$$

We know that Cos (A + B) = Cos A. Cos B – Sin A. Sin B, substituting this we get,

$$\Rightarrow x = \cos\frac{\pi}{4}.\cos\frac{\pi}{6} - \sin\frac{\pi}{4}.\sin\frac{\pi}{6}$$

$$\Rightarrow x = \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}.\frac{1}{2}$$

$\Rightarrow x = \frac{\sqrt{3}-1}{2\sqrt{2}}$

Now, putting the value of $\cos^{-1} x$ in equation (ii)

$\Rightarrow \frac{5\pi}{12} - \cos^{-1} y = \frac{\pi}{6}$

$\Rightarrow \cos^{-1} y = \frac{\pi}{4}$

$\Rightarrow y = \frac{1}{\sqrt{2}}$

$\Rightarrow x = \frac{\sqrt{3}-1}{2\sqrt{2}}$ And $y = \frac{1}{\sqrt{2}}$

4. If cot ($\cos^{-1}$ 3/5 + $\sin^{-1}$ x) = 0, find the value of x.

Solution:
Given cot ($\cos^{-1}$ 3/5 + $\sin^{-1}$ x) = 0
On rearranging we get,
($\cos^{-1}$ 3/5 + $\sin^{-1}$ x) = $\cot^{-1}$ (0)
(Cos^{-1} 3/5 + $\sin^{-1}$ x) = π/2
We know that $\cos^{-1}$ x + $\sin^{-1}$ x = π/2
Then $\sin^{-1}$ x = π/2 - $\cos^{-1}$ x
Substituting the above in ($\cos^{-1}$ 3/5 + $\sin^{-1}$ x) = π/2 we get,
(Cos^{-1} 3/5 + π/2 - $\cos^{-1}$ x) = π/2
Now on rearranging we get,
(Cos^{-1} 3/5 - $\cos^{-1}$ x) = π/2 - π/2
(Cos^{-1} 3/5 - $\cos^{-1}$ x) = 0
Therefore Cos^{-1} 3/5 = $\cos^{-1}$ x
On comparing the above equation we get,

x = 3/5

5. If $(\sin^{-1} x)^2 + (\cos^{-1} x)^2 = 17\pi^2/36$, find x.

Solution:
Given $(\sin^{-1} x)^2 + (\cos^{-1} x)^2 = 17\pi^2/36$
We know that $\cos^{-1} x + \sin^{-1} x = \pi/2$
Then $\cos^{-1} x = \pi/2 - \sin^{-1} x$
Substituting this in $(\sin^{-1} x)^2 + (\cos^{-1} x)^2 = 17\pi^2/36$ we get
$(\sin^{-1} x)^2 + (\pi/2 - \sin^{-1} x)^2 = 17\pi^2/36$
Let $y = \sin^{-1} x$
$y^2 + ((\pi/2) - y)^2 = 17\pi^2/36$
$y^2 + \pi^2/4 - y^2 - 2y((\pi/2) - y) = 17\pi^2/36$
$\pi^2/4 - \pi y + 2y^2 = 17\pi^2/36$
On rearranging and simplifying, we get
$2y^2 - \pi y + 2/9\,\pi^2 = 0$
$18y^2 - 9\pi y + 2\pi^2 = 0$
$18y^2 - 12\pi y + 3\pi y + 2\pi^2 = 0$
$6y(3y - 2\pi) + \pi(3y - 2\pi) = 0$
Now, $(3y - 2\pi) = 0$ and $(6y + \pi) = 0$
Therefore $y = 2\pi/3$ and $y = -\pi/6$
Now substituting $y = -\pi/6$ in $y = \sin^{-1} x$ we get
$\sin^{-1} x = -\pi/6$
$x = \sin(-\pi/6)$
$x = -1/2$
Now substituting $y = -2\pi/3$ in $y = \sin^{-1} x$ we get
$x = \sin(2\pi/3)$
$x = \sqrt{3}/2$
Now substituting $x = \sqrt{3}/2$ in $(\sin^{-1} x)^2 + (\cos^{-1} x)^2 = 17\pi^2/36$ we get,
$= \pi/3 + \pi/6$
$= \pi/2$ which is not equal to $17\pi^2/36$
So we have to neglect this root.
Now substituting $x = -1/2$ in $(\sin^{-1} x)^2 + (\cos^{-1} x)^2 = 17\pi^2/36$ we get,
$= \pi^2/36 + 4\pi^2/9$
$= 17\pi^2/36$
Hence x = -1/2.

EXERCISE 4.11 PAGE NO: 4.82

1. Prove the following results:
(i) $\tan^{-1}(1/7) + \tan^{-1}(1/13) = \tan^{-1}(2/9)$
(ii) $\sin^{-1}(12/13) + \cos^{-1}(4/5) + \tan^{-1}(63/16) = \pi$
(iii) $\tan^{-1}(1/4) + \tan^{-1}(2/9) = \sin^{-1}(1/\sqrt{5})$

Solution:
(i) Given $\tan^{-1}(1/7) + \tan^{-1}(1/13) = \tan^{-1}(2/9)$

Consider LHS

$\tan^{-1}(\frac{1}{7}) + \tan^{-1}(\frac{1}{13})$

We know that, Formula

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$$

According to the formula, we can write as

$$= \tan^{-1}\left(\frac{\frac{1}{7}+\frac{1}{13}}{1-\frac{1}{7}\times\frac{1}{13}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{13+7}{91}}{\frac{91-1}{91}}\right)$$

$$= \tan^{-1}\left(\frac{20}{90}\right)$$

$$= \tan^{-1}\left(\frac{2}{9}\right)$$

= RHS

Hence, the proof.

(ii) Given $\sin^{-1}(12/13) + \cos^{-1}(4/5) + \tan^{-1}(63/16) = \pi$
Consider LHS

$\sin^{-1}(\frac{12}{13}) + \cos^{-1}\frac{4}{5} + \tan^{-1}(\frac{63}{16})$

We know that, Formula

$\sin^{-1}x = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

$\cos^{-1}x = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$

Now, by substituting the formula we get,

$= \tan^{-1}\left(\frac{\frac{12}{13}}{\sqrt{1-\left(\frac{12}{13}\right)^2}}\right) + \tan^{-1}\left(\frac{\sqrt{1-\left(\frac{4}{5}\right)^2}}{\frac{4}{5}}\right) + \tan^{-1}(\frac{63}{16})$

$= \tan^{-1}(\frac{12}{5}) + \tan^{-1}(\frac{3}{4}) + \tan^{-1}(\frac{63}{16})$

Again we know that,

$\tan^{-1}x + \tan^{-1}y = \pi + \tan^{-1}\frac{x+y}{1-xy}$

Again by substituting, we get

$= \pi + \tan^{-1}(\frac{\frac{12}{5}+\frac{3}{4}}{1-\frac{12}{5}\times\frac{3}{4}}) + \tan^{-1}(\frac{63}{16})$

$= \pi + \tan^{-1}(-\frac{63}{16}) + \tan^{-1}(\frac{63}{16})$

We know that,

$\tan^{-1}(-x) = -\tan^{-1}x$

$= \pi - \tan^{-1}(-\frac{63}{16}) + \tan^{-1}(\frac{63}{16})$

$= \pi$

So, $\sin^{-1}(\frac{12}{13}) + \cos^{-1}\frac{4}{5} + \tan^{-1}(\frac{63}{16}) = \pi$

Hence, the proof.

(iii) Given $\tan^{-1}(1/4) + \tan^{-1}(2/9) = \sin^{-1}(1/\sqrt{5})$

$\tan^{-1}(\frac{1}{4}) + \tan^{-1}(\frac{2}{9})$

We know that,

$\tan^{-1}x + \tan^{-1}y = \pi + \tan^{-1}\frac{x+y}{1-xy}$

By substituting this formula we get,

$= \tan^{-1}\frac{\frac{1}{4}+\frac{2}{9}}{1-\frac{1}{4}\times\frac{2}{9}}$

$= \tan^{-1}\frac{\frac{17}{36}}{\frac{34}{36}}$

$= \tan^{-1}\frac{\frac{17}{36}}{\frac{34}{36}}$

$= \tan^{-1}\frac{1}{2}$

Now let, $\tan\theta = \frac{1}{2}$

Therefore, $\sin\theta = \frac{1}{\sqrt{5}}$

So, $\theta = \sin^{-1}\frac{1}{\sqrt{5}}$

$\Rightarrow \tan^{-1}(\frac{1}{2}) = \sin^{-1}(\frac{1}{\sqrt{5}}) = \text{RHS}$

$\tan^{-1}(\frac{1}{4}) + \tan^{-1}(\frac{2}{9}) = \sin^{-1}(\frac{1}{\sqrt{5}})$

Hence, Proved.

2. Find the value of $\tan^{-1}$ (x/y) – $\tan^{-1}$ {(x-y)/(x + y)}

Solution:

Given $\tan^{-1}$ (x/y) – $\tan^{-1}$ {(x-y)/(x + y)}

We know that,

$$\tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x - y}{1 + xy}$$

Now by substituting the formula, we get

$= \tan^{-1} \frac{\frac{x}{y} - \left(\frac{x-y}{x+y}\right)}{1 + \frac{x}{y} \times \left(\frac{x-y}{x+y}\right)}$

$= \tan^{-1} \frac{x(x+y) - y(x-y)}{y(x+y) + x(x-y)}$

$= \tan^{-1} \frac{x^2 + y^2}{x^2 + y^2}$

$= \tan^{-1} 1$

$= \frac{\pi}{4}$

So,

$$\tan^{-1}(\frac{x}{y}) - \tan^{-1}(\frac{x - y}{x + y}) = \frac{\pi}{4}$$

EXERCISE 4.12

1. Evaluate: Cos (sin $^{-1}$ 3/5 + sin^{-1} 5/13)

Solution:

Given Cos (sin $^{-1}$ 3/5 + sin^{-1} 5/13)

We know that,

$$\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$$

By substituting this formula we get,

$$= \cos\left(\sin^{-1}\left[\frac{3}{5}\sqrt{1-\left(\frac{5}{13}\right)^2} + \frac{5}{13}\sqrt{1-\left(\frac{3}{5}\right)^2}\right]\right)$$

$$= \cos\left(\sin^{-1}\left[\frac{3}{5}\times\frac{12}{13} + \frac{5}{13}\times\frac{4}{5}\right]\right)$$

$$= \cos\left(\sin^{-1}\left[\frac{56}{65}\right]\right)$$

Again, we know that

$$\sin^{-1}x = \cos^{-1}\sqrt{1-x^2}$$

Now substituting, we get

$$= \cos\left(\cos^{-1}\sqrt{1-\left(\frac{56}{65}\right)^2}\right)$$

$$= \cos\left(\cos^{-1}\sqrt{\frac{33}{65}}\right)$$

$$= \frac{33}{65}$$

Hence, $\cos\left(\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{5}{13}\right) = \frac{33}{65}$

EXERCISE 4.13 PAGE NO: 4.92

1. If $\cos^{-1}(x/2) + \cos^{-1}(y/3) = \alpha$, then prove that $9x^2 - 12xy\cos\alpha + 4y^2 = 36\sin^2\alpha$

Solution:

Given $\cos^{-1}(x/2) + \cos^{-1}(y/3) = \alpha$

We know that,

$$\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left[xy - \sqrt{1-x^2}\sqrt{1-y^2}\right]$$

Now by substituting, we get

$$\Rightarrow \cos^{-1}\left[\frac{x}{2}\times\frac{y}{3} - \sqrt{1-\left(\frac{x}{2}\right)^2}\sqrt{1-\left(\frac{y}{3}\right)^2}\right] = \alpha$$

$$\Rightarrow \left[\frac{xy}{6} - \frac{\sqrt{4-x^2}}{2}\times\frac{\sqrt{9-y^2}}{3}\right] = \cos\alpha$$

$$\Rightarrow xy - \sqrt{4-x^2}\times\sqrt{9-y^2} = 6\cos\alpha$$

$$\Rightarrow xy - 6\cos\alpha = \sqrt{4-x^2}\sqrt{9-y^2}$$

On squaring both the sides we get

$$\Rightarrow (xy - 6\cos\alpha)^2 = (4-x^2)(9-y^2)$$

$$\Rightarrow x^2y^2 + 36\cos^2\alpha - 12xy\cos\alpha = 36 - 9x^2 - 4y^2 + x^2y^2$$

$$\Rightarrow 9x^2 + 4y^2 - 36 + 36\cos^2\alpha - 12xy\cos\alpha = 0$$

$\Rightarrow 9x^2 + 4y^2 - 12xy\cos\alpha - 36(1 - \cos^2\alpha) = 0$

$\Rightarrow 9x^2 + 4y^2 - 12xy\cos\alpha - 36\sin^2\alpha = 0$

$\Rightarrow 9x^2 + 4y^2 - 12xy\cos\alpha = 36\sin^2\alpha$

Hence the proof.

2. Solve the equation: $\cos^{-1}(a/x) - \cos^{-1}(b/x) = \cos^{-1}(1/b) - \cos^{-1}(1/a)$

Solution:

Given $\cos^{-1}(a/x) - \cos^{-1}(b/x) = \cos^{-1}(1/b) - \cos^{-1}(1/a)$

$\Rightarrow \cos^{-1}\frac{a}{x} + \cos^{-1}\frac{1}{a} = \cos^{-1}\frac{1}{b} + \cos^{-1}\frac{b}{x}$

We know that,

$\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left[xy - \sqrt{1-x^2}\sqrt{1-y^2}\right]$

By substituting this formula we get,

$\Rightarrow \cos^{-1}\left[\frac{1}{x} - \sqrt{1-\left(\frac{a}{x}\right)^2}\sqrt{1-\left(\frac{1}{a}\right)^2}\right] = \cos^{-1}\left[\frac{1}{x} - \sqrt{1-\left(\frac{b}{x}\right)^2}\sqrt{1-\left(\frac{1}{b}\right)^2}\right]$

$\Rightarrow \frac{1}{x} - \sqrt{1-\left(\frac{a}{x}\right)^2}\sqrt{1-\left(\frac{1}{a}\right)^2} = \frac{1}{x} - \sqrt{1-\left(\frac{b}{x}\right)^2}\sqrt{1-\left(\frac{1}{b}\right)^2}$

$\Rightarrow \sqrt{1-\left(\frac{a}{x}\right)^2}\sqrt{1-\left(\frac{1}{a}\right)^2} = \sqrt{1-\left(\frac{b}{x}\right)^2}\sqrt{1-\left(\frac{1}{b}\right)^2}$

Squaring on both the sides, we get

$\Rightarrow \left(1-\left(\frac{a}{x}\right)^2\right)\left(1-\left(\frac{1}{a}\right)^2\right) = \left(1-\left(\frac{b}{x}\right)^2\right)\left(1-\left(\frac{1}{b}\right)^2\right)$

$\Rightarrow 1-\left(\frac{a}{x}\right)^2-\left(\frac{1}{a}\right)^2+\left(\frac{1}{x}\right)^2 = 1-\left(\frac{b}{x}\right)^2-\left(\frac{1}{b}\right)^2+\left(\frac{1}{x}\right)^2$

$\Rightarrow \left(\frac{b}{x}\right)^2-\left(\frac{a}{x}\right)^2=\left(\frac{1}{a}\right)^2-\left(\frac{1}{b}\right)^2$

On simplifying, we get

$\Rightarrow (b^2 - a^2)\ a^2b^2 = x^2(b^2 - a^2)$

$\Rightarrow x^2 = a^2b^2$

$\Rightarrow x = a\ b$

EXERCISE 4.14

1. Evaluate the following:
(i) tan {2 tan^{-1} (1/5) – π/4}
(ii) Tan {1/2 sin^{-1} (3/4)}
(iii) Sin {1/2 cos^{-1} (4/5)}
(iv) Sin (2 tan $^{-1}$ 2/3) + cos (tan^{-1} √3)

Solution:
(i) Given tan {2 tan^{-1} (1/5) – π/4}

We know that,

$$2\tan^{-1}(x) = \tan^{-1}\left(\frac{2x}{1-x^2}\right), \text{if } |x| < 1$$

And $\frac{\pi}{4}$ can be written as $\tan^{-1}(1)$

Now substituting these values we get,

$$= \tan\left\{\tan^{-1}\left(\frac{2\times\frac{1}{5}}{1-\frac{1}{25}}\right) - \tan^{-1}1\right\}$$

$$= \tan\left\{\tan^{-1}\left(\frac{5}{12}\right) - \tan^{-1}1\right\}$$

Again we know that,

$$\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}$$

Now substituting this formula, we get

$$= \tan\left\{\tan^{-1}\left(\frac{\frac{5}{12}-1}{1+\frac{5}{12}}\right)\right\}$$

$= \tan\left\{\tan^{-1}(\frac{-7}{17})\right\}$

$= -\frac{7}{17}$

(ii) Given tan {1/2 $\sin^{-1}$ (3/4)}

Let $\frac{1}{2}\sin^{-1}\frac{3}{4} = t$

Therefore,

$\Rightarrow \sin^{-1}\frac{3}{4} = 2t$

$\Rightarrow \sin 2t = \frac{3}{4}$

Now, by Pythagoras theorem, we have

$\Rightarrow \sin 2t = \frac{3}{4} = \frac{\text{perpendicular}}{\text{hypotenuse}}$

$\Rightarrow \cos 2t = \frac{\sqrt{4^2 - 3^2}}{4} = \frac{\text{Base}}{\text{hypotenuse}}$

$\Rightarrow \cos 2t = \frac{\sqrt{7}}{4}$

By considering, given question

$\tan\left\{\frac{1}{2}\sin^{-1}\frac{3}{4}\right\}$

$= \tan(t)$

We know that,

$$\tan(x) = \sqrt{\frac{1-\cos 2x}{1+\cos 2x}}$$

$$= \sqrt{\frac{1-\cos 2t}{1+\cos 2t}}$$

$$= \sqrt{\frac{1-\frac{\sqrt{7}}{4}}{1+\frac{\sqrt{7}}{4}}}$$

$$= \sqrt{\frac{4-\sqrt{7}}{4+\sqrt{7}}}$$

Now by rationalizing the denominator, we get

$$= \sqrt{\frac{(4-\sqrt{7})(4-\sqrt{7})}{(4+\sqrt{7})(4-\sqrt{7})}}$$

$$= \sqrt{\frac{(4-\sqrt{7})^2}{9}}$$

$$= \frac{4-\sqrt{7}}{3}$$

Hence

$$\tan\left\{\frac{1}{2}\sin^{-1}\frac{3}{4}\right\} = \frac{4-\sqrt{7}}{3}$$

(iii) Given sin {1/2 $\cos^{-1}$ (4/5)}

We know that

$$\cos^{-1}x = 2\sin^{-1}\left(\pm\sqrt{\frac{1-x}{2}}\right)$$

Now by substituting this formula we get,

$$\sin\left(\frac{1}{2}2\sin^{-1}\left(\pm\sqrt{\frac{1-\frac{4}{5}}{2}}\right)\right)$$

$= \sin\left(\sin^{-1}\left(\pm\sqrt{\frac{1}{2\times5}}\right)\right)$

$= \sin\left(\sin^{-1}\left(\pm\frac{1}{\sqrt{10}}\right)\right)$

As we know that

$\sin(\sin^{-1}x) = x \text{ as } n \in [-1, 1]$

$= \pm\frac{1}{\sqrt{10}}$

Hence, $\sin\left(\frac{1}{2}\cos^{-1}\frac{4}{5}\right) = \pm\frac{1}{\sqrt{10}}$

(iv) Given Sin (2 tan $^{-1}$ 2/3) + cos (tan^{-1} √3)

We know that

$\sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\tan^{-1}(x)$;

$\cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) = \tan^{-1}(x)$;

Now by substituting these formulae we get,

$= \sin\left(\sin^{-1}\left(\frac{2\times\frac{2}{3}}{1+\frac{4}{9}}\right)\right) + \cos\left(\cos^{-1}\left(\frac{1}{\sqrt{1+3}}\right)\right)$

$= \sin\left(\sin^{-1}\left(\frac{12}{13}\right)\right) + \cos\left(\cos^{-1}\left(\frac{1}{2}\right)\right)$

$= \frac{12}{13} + \frac{1}{2}$

$= \frac{37}{26}$

Hence,

$\sin\left(2\tan^{-1}\left(\frac{2}{3}\right)\right) + \cos(\tan^{-1}\sqrt{3}) = \frac{37}{26}$

2. Prove the following results:
(i) $2\sin^{-1}(3/5) = \tan^{-1}(24/7)$
(ii) $\tan^{-1}(1/4) + \tan^{-1}(2/9) = \frac{1}{2}\cos^{-1}(3/5) = \frac{1}{2}\sin^{-1}(4/5)$
(iii) $\tan^{-1}(2/3) = \frac{1}{2}\tan^{-1}(12/5)$
(iv) $\tan^{-1}(1/7) + 2\tan^{-1}(1/3) = \pi/4$
(v) $\sin^{-1}(4/5) + 2\tan^{-1}(1/3) = \pi/2$
(vi) $2\sin^{-1}(3/5) - \tan^{-1}(17/31) = \pi/4$
(vii) $2\tan^{-1}(1/5) + \tan^{-1}(1/8) = \tan^{-1}(4/7)$
(viii) $2\tan^{-1}(3/4) - \tan^{-1}(17/31) = \pi/4$
(ix) $2\tan^{-1}(1/2) + \tan^{-1}(1/7) = \tan^{-1}(31/17)$
(x) $4\tan^{-1}(1/5) - \tan^{-1}(1/239) = \pi/4$

Solution:
(i) Given $2\sin^{-1}(3/5) = \tan^{-1}(24/7)$

Consider LHS

$2\sin^{-1}\frac{3}{5}$

We know that

$\sin^{-1}(x) = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

Now by substituting the above formula we get,

$= 2 \times \tan^{-1}\left(\frac{\frac{3}{5}}{\sqrt{1-\frac{9}{25}}}\right)$

$= 2 \times \tan^{-1}\left(\frac{\frac{3}{5}}{\frac{4}{5}}\right)$

$= 2 \times \tan^{-1}\left(\frac{3}{4}\right)$

Again we know that

$$2\tan^{-1}(x) = \tan^{-1}\left(\frac{2x}{1-x^2}\right), \text{if } |x| < 1$$

Therefore,

$= \tan^{-1}\left(\frac{2\times\frac{3}{4}}{1-\frac{9}{16}}\right)$

$= \tan^{-1}\left(\frac{\frac{3}{2}}{\frac{7}{16}}\right)$

$= \tan^{-1}\left(\frac{24}{7}\right)$

= RHS

So, $2\sin^{-1}\frac{3}{5} = \tan^{-1}\left(\frac{24}{7}\right)$

Hence the proof.

(ii) Given $\tan^{-1}$ ¼ + $\tan^{-1}$ (2/9) = ½ $\cos^{-1}$ (3/5) = ½ $\sin^{-1}$ (4/5)

Consider LHS

$= \tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{2}{9}\right)$

We know that

$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}$

Now by substituting this formula, we get

$= \tan^{-1}\left(\frac{\frac{1}{4}+\frac{2}{9}}{1-\frac{1}{4}\times\frac{2}{9}}\right)$

$= \tan^{-1}\left(\frac{\frac{9+8}{36}}{\frac{36-2}{36}}\right)$

$= \tan^{-1}\left(\frac{17}{34}\right)$

$= \tan^{-1}\left(\frac{1}{2}\right)$

Multiplying and dividing by 2

$= \frac{1}{2}\left\{2\tan^{-1}\left(\frac{1}{2}\right)\right\}$

Again we know that

$2\tan^{-1} x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$= \frac{1}{2}\cos^{-1}\left(\frac{1-\frac{1}{4}}{1+\frac{1}{4}}\right)$

$= \frac{1}{2}\cos^{-1}\left(\frac{\frac{3}{4}}{\frac{5}{4}}\right)$

$= \frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right)$

= RHS

So, $\tan^{-1}(\frac{1}{4}) + \tan^{-1}(\frac{2}{9}) = \frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right)$

Now,

$= \frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right)$

We know that,

$= \cos^{-1}x = \sin^{-1}\sqrt{1-x^2}$

By substituting this, we get

$= \frac{1}{2}\sin^{-1}\sqrt{1-\frac{9}{25}}$

$= \frac{1}{2}\sin^{-1}\sqrt{\frac{16}{25}}$

$= \frac{1}{2}\sin^{-1}\frac{4}{5}$

= RHS

So, $\tan^{-1}(\frac{1}{4}) + \tan^{-1}(\frac{2}{9}) = \frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right) = \frac{1}{2}\sin^{-1}\frac{4}{5}$

Hence the proof.

(iii) Given $\tan^{-1}(2/3) = ½ \tan^{-1}(12/5)$

Consider LHS

$= \tan^{-1}(\frac{2}{3})$

Now, Multiplying and dividing by 2, we get

$= \frac{1}{2}\left\{2\tan^{-1}\left(\frac{2}{3}\right)\right\}$

We know that

$2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

By substituting the above formula we get

$= \frac{1}{2}\tan^{-1}\left(\frac{2\times\frac{2}{3}}{1-\frac{4}{9}}\right)$

$= \frac{1}{2}\tan^{-1}\left(\frac{\frac{4}{3}}{\frac{5}{9}}\right)$

$= \frac{1}{2}tan^{-1}\left(\frac{12}{5}\right)$

= RHS

So, $tan^{-1}(\frac{2}{3}) = \frac{1}{2}tan^{-1}\left(\frac{12}{5}\right)$

Hence the proof.

(iv) Given $\tan^{-1}(1/7) + 2\tan^{-1}(1/3) = \pi/4$

Consider LHS

$= \tan^{-1}(\frac{1}{7}) + 2\tan^{-1}(\frac{1}{3})$

We know that,

$2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

By substituting the above formula we get,

$= \tan^{-1}(\frac{1}{7}) + \tan^{-1}(\frac{2\times\frac{1}{3}}{1-\frac{1}{9}})$

= $\tan^{-1}(\frac{1}{7}) + \tan^{-1}(\frac{\frac{2}{3}}{\frac{8}{9}})$

= $\tan^{-1}(\frac{1}{7}) + \tan^{-1}(\frac{3}{4})$

Again we know that

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$$

= $\tan^{-1}\left(\frac{\frac{1}{7}+\frac{3}{4}}{1-\frac{1}{7}\times\frac{3}{4}}\right)$

= $\tan^{-1}\left(\frac{\frac{25}{28}}{\frac{25}{28}}\right)$

= $\tan^{-1}(1)$

= $\frac{\pi}{4}$

= RHS

So, $\tan^{-1}(\frac{1}{7}) + 2\tan^{-1}(\frac{1}{3}) = \frac{\pi}{4}$

Hence the proof.

(v) Given $\sin^{-1}(4/5) + 2\tan^{-1}(1/3) = \pi/2$

Consider LHS

= $\sin^{-1}(\frac{4}{5}) + 2\tan^{-1}(\frac{1}{3})$

We know that,

$\sin^{-1}(x) = \tan^{-1}(\frac{x}{\sqrt{1-x^2}})$

And, $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

Now by substituting the formula we get,

$= \tan^{-1}(\frac{\frac{4}{5}}{\sqrt{1-\frac{16}{25}}}) + \tan^{-1}(\frac{2\times\frac{1}{3}}{1-\frac{1}{9}})$

$= \tan^{-1}(\frac{\frac{4}{5}}{\sqrt{\frac{9}{25}}}) + \tan^{-1}(\frac{\frac{2}{3}}{\frac{8}{9}})$

$= \tan^{-1}(\frac{4}{3}) + \tan^{-1}(\frac{3}{4})$

We know that,

$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$

$= \tan^{-1}\left(\frac{\frac{4}{3}+\frac{3}{4}}{1-\frac{4}{3}\times\frac{3}{4}}\right)$

$= \tan^{-1}\left(\frac{\frac{25}{12}}{0}\right)$

$= \tan^{-1}(\infty)$

$= \frac{\pi}{2}$

= RHS

So, $\sin^{-1}(\frac{4}{5}) + 2\tan^{-1}(\frac{1}{3}) = \frac{\pi}{2}$

Hence Proved

(vi) Given 2 $\sin^{-1}$ (3/5) – $\tan^{-1}$ (17/31) = π/4

Consider LHS

$= 2\sin^{-1}(\frac{3}{5}) - \tan^{-1}(\frac{17}{31})$

We know that

$$\sin^{-1}(x) = \tan^{-1}(\frac{x}{\sqrt{1-x^2}})$$

According to the formula we have,

$$= 2\tan^{-1}(\frac{\frac{3}{5}}{\sqrt{1-\frac{16}{25}}}) - \tan^{-1}(\frac{17}{31})$$

$$= 2\tan^{-1}(\frac{\frac{4}{5}}{\sqrt{\frac{9}{25}}}) - \tan^{-1}(\frac{17}{31})$$

$$= 2\tan^{-1}(\frac{3}{4}) - \tan^{-1}(\frac{17}{31})$$

Again we know that,

$$2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

By substituting this formula, we get

$$= \tan^{-1}(\frac{2\times\frac{3}{4}}{1-\frac{9}{16}}) - \tan^{-1}(\frac{17}{31})$$

$= \tan^{-1}(\frac{\frac{3}{\frac{2}{7}}}{16}) - \tan^{-1}(\frac{17}{31})$

$= \tan^{-1}(\frac{24}{7}) - \tan^{-1}(\frac{17}{31})$

Again we have,

$$\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}$$

$= \tan^{-1}\left(\frac{\frac{24}{7}-\frac{17}{31}}{1+\frac{24}{7}\times\frac{17}{31}}\right)$

$= \tan^{-1}\left(\frac{\frac{744-119}{217}}{\frac{217+408}{217}}\right)$

$= \tan^{-1}\left(\frac{625}{625}\right)$

$= \tan^{-1}(1)$

$= \frac{\pi}{4}$ = RHS

So, $2\sin^{-1}(\frac{3}{5}) - \tan^{-1}(\frac{17}{31}) = \frac{\pi}{4}$

Hence the proof.

(vii) Given 2 $\tan^{-1}$ (1/5) + $\tan^{-1}$ (1/8) = $\tan^{-1}$ (4/7)

Consider LHS

$= 2\tan^{-1}(\frac{1}{5}) + \tan^{-1}(\frac{1}{8})$

We know that

$2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

Now by substituting the formula we get,

$= \tan^{-1}\left(\frac{2\times\frac{1}{5}}{1-\frac{1}{25}}\right) + \tan^{-1}\left(\frac{1}{8}\right)$

$= \tan^{-1}\left(\frac{\frac{2}{5}}{\frac{24}{25}}\right) + \tan^{-1}\left(\frac{1}{8}\right)$

$= \tan^{-1}\left(\frac{5}{12}\right) + \tan^{-1}\left(\frac{1}{8}\right)$

Again from the formula we have,

$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$

$= \tan^{-1}\left(\frac{\frac{5}{12}+\frac{1}{8}}{1-\frac{5}{12}\times\frac{1}{8}}\right)$

$= \tan^{-1}\left(\frac{\frac{10+3}{24}}{\frac{96-5}{96}}\right)$

$= \tan^{-1}\left(\frac{13}{24}\times\frac{96}{91}\right)$

$= \tan^{-1}\left(\frac{4}{7}\right)$

= RHS

So, $2\tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \tan^{-1}\left(\frac{4}{7}\right)$

Hence the proof.

(viii) Given 2 $\tan^{-1}$ (3/4) - $\tan^{-1}$ (17/31) = π/4

Consider LHS

$= 2\tan^{-1}(\frac{3}{4}) - \tan^{-1}(\frac{17}{31})$

We know that,

$$2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

Now by substituting the formula we get,

$= \tan^{-1}(\frac{2\times\frac{3}{4}}{1-\frac{9}{16}}) - \tan^{-1}(\frac{17}{31})$

$= \tan^{-1}(\frac{3}{2}\times\frac{16}{7}) - \tan^{-1}(\frac{17}{31})$

$= \tan^{-1}(\frac{24}{7}) - \tan^{-1}(\frac{17}{31})$

We know that,

$$\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}$$

Again by substituting the formula we get,

$= \tan^{-1}\left(\frac{\frac{24}{7}-\frac{17}{31}}{1+\frac{24}{7}\times\frac{17}{31}}\right)$

$= \tan^{-1}\left(\frac{\frac{744-119}{217}}{\frac{217+408}{217}}\right)$

$= \tan^{-1}\left(\frac{625}{625}\right)$

$= \tan^{-1}(1)$

$= \frac{\pi}{4}$

= RHS

So, $2\tan^{-1}(\frac{3}{4}) - \tan^{-1}(\frac{17}{31}) = \frac{\pi}{4}$

Hence the proof.

(ix) Given 2 $\tan^{-1}$ (1/2) + $\tan^{-1}$ (1/7) = $\tan^{-1}$ (31/17)

Consider LHS

= $2\tan^{-1}(\frac{1}{2}) + \tan^{-1}(\frac{1}{7})$

We know that,

$$2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

Now by substituting the formula we get,

= $\tan^{-1}(\frac{2\times\frac{1}{2}}{1-\frac{1}{4}}) + \tan^{-1}(\frac{1}{7})$

= $\tan^{-1}(\frac{\frac{2}{2}}{\frac{3}{4}}) + \tan^{-1}(\frac{1}{7})$

= $\tan^{-1}(\frac{4}{3}) + \tan^{-1}(\frac{1}{7})$

Again by using the formula, we can write as

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$$

= $\tan^{-1}\left(\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{1}{7}\times\frac{4}{3}}\right)$

$= \tan^{-1}\left(\frac{\frac{31}{21}}{\frac{17}{21}}\right)$

$= \tan^{-1}\left(\frac{31}{17}\right)$

= RHS

So, $2\tan^{-1}(\frac{1}{2}) + \tan^{-1}(\frac{1}{7}) = \tan^{-1}(\frac{31}{17})$

Hence the proof.

(x) Given 4 $\tan^{-1}(1/5) - \tan^{-1}(1/239) = \pi/4$

Consider LHS

$= 4\tan^{-1}(\frac{1}{5}) - \tan^{-1}(\frac{1}{239})$

We know that,

$4\tan^{-1}x = \tan^{-1}\left(\frac{4x - 4x^3}{1 - 6x^2 + x^4}\right)$

Now by substituting the formula, we get

$= \tan^{-1}\left(\frac{4\times\frac{1}{5} - 4\left(\frac{1}{5}\right)^3}{1 - 6\left(\frac{1}{5}\right)^2 + \left(\frac{1}{5}\right)^4}\right) - \tan^{-1}(\frac{1}{239})$

$= \tan^{-1}(\frac{120}{119}) - \tan^{-1}(\frac{1}{239})$

Again we know that,

$\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x - y}{1 + xy}$

$= \tan^{-1}\left(\frac{\frac{120}{119} - \frac{1}{239}}{1 - \frac{120}{119}\times\frac{1}{239}}\right)$

$= \tan^{-1}\left(\frac{120\times239-119}{119\times239+120}\right)$

$= \tan^{-1}\left(\frac{28561}{28561}\right)$

$= \tan^{-1}(1)$

$= \frac{\pi}{4}$

= RHS

So,

$4\tan^{-1}(\frac{1}{5}) - \tan^{-1}(\frac{1}{239}) = \frac{\pi}{4}$

Hence the proof.

3. If $\sin^{-1}(2a/1 + a^2) - \cos^{-1}(1 - b^2/1 + b^2) = \tan^{-1}(2x/1 - x^2)$, then prove that $x = (a - b)/(1 + a b)$

Solution:

Given $\sin^{-1}(2a/1 + a^2) - \cos^{-1}(1 - b^2/1 + b^2) = \tan^{-1}(2x/1 - x^2)$

Consider,

$\Rightarrow \sin^{-1}(\frac{2a}{1+a^2}) - \cos^{-1}\frac{1-b^2}{1+b^2} = \tan^{-1}(\frac{2x}{1-x^2})$

We know that,

$2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$2\tan^{-1}x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

Now by applying these formulae in given equation we get,

$\Rightarrow 2\tan^{-1}(a) - 2\tan^{-1}(b) = 2\tan^{-1}(x)$

$\Rightarrow 2(\tan^{-1}(a) - \tan^{-1}(b)) = 2\tan^{-1}(x)$

$\Rightarrow \tan^{-1}(a) - \tan^{-1}(b) = \tan^{-1}(x)$

Again we know that,

$$\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}$$

Now by substituting this in above equation we get,

$\Rightarrow \tan^{-1}(\frac{a-b}{1+ab}) = \tan^{-1}(x)$

On comparing we get,

$\Rightarrow x = \frac{a-b}{1+ab}$

Hence the proof.

4. Prove that:
(i) $\tan^{-1}\{(1 - x^2)/ 2x)\} + \cot^{-1}\{(1 - x^2)/ 2x)\} = \pi/2$
(ii) $\sin \{\tan^{-1} (1 - x^2)/ 2x) + \cos^{-1} (1 - x^2)/ (1 + x^2)\} = 1$

Solution:
(i) Given $\tan^{-1}\{(1 - x^2)/ 2x)\} + \cot^{-1}\{(1 - x^2)/ 2x)\} = \pi/2$
Consider LHS

$= \tan^{-1}\frac{1-x^2}{2x} + \cot^{-1}\frac{1-x^2}{2x}$

We know that,

$$\cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right)$$

Now by applying the above formula we get,

$= \tan^{-1}\left(\frac{1-x^2}{2x}\right) + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

Again we know

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\frac{x+y}{1-xy}$$

By substituting this we get,

$= \tan^{-1}\left(\frac{\left(\frac{1-x^2}{2x}\right)+\left(\frac{2x}{1-x^2}\right)}{1-\left(\frac{1-x^2}{2x}\right)\times\left(\frac{2x}{1-x^2}\right)}\right)$

$= \tan^{-1}\left(\frac{\frac{1+x^4-2x^2+4x^2}{2x(1-x^2)}}{\frac{2x(1-x^2)-2x(1-x^2)}{2x(1-x^2)}}\right)$

$= \tan^{-1}\left(\frac{1+x^4+2x^2}{0}\right)$

$= \tan^{-1}(\infty)$

$= \frac{\pi}{2}$ = RHS

$$\tan^{-1}\frac{1-x^2}{2x} + \cot^{-1}\frac{1-x^2}{2x} = \frac{\pi}{2}$$

(ii) Given sin {$\tan^{-1} (1 - x^2)/ 2x) + \cos^{-1} (1 - x^2)/ (1 + x^2)$}

Consider LHS

$= \sin\left(\tan^{-1}\frac{1-x^2}{2x} + \cos^{-1}\frac{1-x^2}{1+x^2}\right)$

We know that,

$$2\tan^{-1}x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

Now by applying the formula in above question we get,

$$= \sin\left(\tan^{-1}\frac{1-x^2}{2x} + 2\tan^{-1}x\right)$$

Again, we have

$$2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

Now by substituting the formula we get,

$$= \sin\left(\tan^{-1}\frac{1-x^2}{2x} + \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right)$$

Again we know that,

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$$

Now by applying the formula,

$$= \sin\left(\tan^{-1}\left(\frac{\frac{1-x^2}{2x}+\left(\frac{2x}{1-x^2}\right)}{1-\frac{1-x^2}{2x}\times\left(\frac{2x}{1-x^2}\right)}\right)\right)$$

$$= \sin\left(\tan^{-1}\left(\frac{\frac{1+x^4-2x^2+4x^2}{2x(1-x^2)}}{\frac{2x(1-x^2)-2x(1-x^2)}{2x(1-x^2)}}\right)\right)$$

$$= \sin\left(\tan^{-1}\left(\frac{\frac{1+x^4-2x^2+4x^2}{2x(1-x^2)}}{0}\right)\right)$$

$$= \sin(\tan^{-1}(\infty))$$

$= \sin\left(\frac{\pi}{2}\right)$

= 1

= RHS

So,

$$\sin^{-1}\left(\tan^{-1}\frac{1-x^2}{2x} + \cos^{-1}\frac{1-x^2}{1+x^2}\right) = 1$$

Hence the proof.

5. If $\sin^{-1}(2a/ 1+ a^2) + \sin^{-1}(2b/ 1+ b^2) = 2\tan^{-1} x$, prove that $x = (a + b/ 1 – a b)$

Solution:

Given $\sin^{-1}(2a/ 1+ a^2) + \sin^{-1}(2b/ 1+ b^2) = 2\tan^{-1} x$

Consider

$$\sin^{-1}\left(\frac{2a}{1+a^2}\right) + \sin^{-1}\frac{2b}{1+b^2} = 2\tan^{-1}(x)$$

We know that,

$$2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

Now by applying the above formula we get,

$$\Rightarrow 2\tan^{-1}(a) + 2\tan^{-1}(b) = 2\tan^{-1}(x)$$

$$\Rightarrow 2(\tan^{-1}(a) + \tan^{-1}(b)) = 2\tan^{-1}(x)$$

$$\Rightarrow \tan^{-1}(a) + \tan^{-1}(b) = \tan^{-1}(x)$$

Again we have,

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x + y}{1 - xy}$$

Now by substituting, we get

$\Rightarrow \tan^{-1}(\frac{a+b}{1-ab}) = \tan^{-1}(x)$

On comparing we get,

$\Rightarrow x = \frac{a+b}{1-ab}$

Hence the proof.