EXERCISE 3.1 PAGE NO: 3.4

1. Determine whether the following operation define a binary operation on the given set or not:
(i) '*' on N defined by a * b = a^b for all a, b ∈ N.
(ii) 'O' on Z defined by a O b = a^b for all a, b ∈ Z.
(iii) '*' on N defined by a * b = a + b - 2 for all a, b ∈ N
(iv) '$\times_6$' on S = {1, 2, 3, 4, 5} defined by a $\times_6$ b = Remainder when a b is divided by 6.
(v) '$+_6$' on S = {0, 1, 2, 3, 4, 5} defined by a $+_6$ b

$$= \begin{cases} a+b, \ if \ a+b<6 \\ a+b-6, \ if \ a+b \geq 6 \end{cases}$$

(vi) '⊙' on N defined by a ⊙ b= $a^b + b^a$ for all a, b ∈ N
(vii) '*' on Q defined by a * b = (a – 1)/ (b + 1) for all a, b ∈ Q

Solution:
(i) Given '*' on N defined by a * b = a^b for all a, b ∈ N.
Let a, b ∈ N. Then,
a^b ∈ N [∵ $a^b \neq 0$ and a, b is positive integer]
⇒ a * b ∈ N
Therefore,
a * b ∈ N, ∀ a, b ∈ N
Thus, * is a binary operation on N.

(ii) Given 'O' on Z defined by a O b = a^b for all a, b ∈ Z.
Both a = 3 and b = -1 belong to Z.
⇒ a * b = 3^{-1}
= 1/3 ∉ Z
Thus, * is not a binary operation on Z.

(iii) Given '*' on N defined by a * b = a + b - 2 for all a, b ∈ N
If a = 1 and b = 1,
a * b = a + b - 2
= 1 + 1 - 2
= 0 ∉ N
Thus, there exist a = 1 and b = 1 such that a * b ∉ N
So, * is not a binary operation on N.

(iv) Given '$\times_6$' on S = {1, 2, 3, 4, 5} defined by a $\times_6$ b = Remainder when a b is divided by 6.
Consider the composition table,

X_6	1	2	3	4	5
1	1	2	3	4	5
2	2	4	0	2	4
3	3	0	3	0	3
4	4	2	0	4	2
5	5	4	3	2	1

Here all the elements of the table are not in S.
⇒ For a = 2 and b = 3,
a $\times_6$ b = 2 $\times_6$ 3 = remainder when 6 divided by 6 = 0 ≠ S
Thus, $\times_6$ is not a binary operation on S.

(v) Given '$+_6$' on S = {0, 1, 2, 3, 4, 5} defined by a $+_6$ b

$$= \begin{cases} a+b, \ if \ a+b<6 \\ a+b-6, \ if \ a+b \geq 6 \end{cases}$$

Consider the composition table,

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Here all the elements of the table are not in S.
⇒ For a = 2 and b = 3,

$a \times_6 b = 2 \times_6 3$ = remainder when 6 divided by 6 = 0 ≠
Thus, $\times_6$ is not a binary operation on S.

(vi) Given '⊙' on N defined by a ⊙ b= $a^b + b^a$ for all a, b ∈ N
Let a, b ∈ N. Then,
$a^b, b^a \in N$
⇒ $a^b + b^a \in N$ [∵Addition is binary operation on N]
⇒ a ⊙ b ∈ N
Thus, ⊙ is a binary operation on N.

(vii) Given '*' on Q defined by a * b = (a – 1)/ (b + 1) for all a, b ∈ Q
If a = 2 and b = -1 in Q,
a * b = (a – 1)/ (b + 1)
= (2 – 1)/ (- 1 + 1)
= 1/0 [which is not defined]
For a = 2 and b = -1
a * b does not belongs to Q
So, * is not a binary operation in Q.

2. Determine whether or not the definition of * given below gives a binary operation. In the event that * is not a binary operation give justification of this.
(i) On Z^+, defined * by a * b = a – b
(ii) On Z^+, define * by a*b = ab
(iii) On R, define * by a*b = ab^2
(iv) On Z^+ define * by a * b = |a – b|
(v) On Z^+ define * by a * b = a
(vi) On R, define * by a * b = a + $4b^2$
Here, Z^+ denotes the set of all non-negative integers.

Solution:
(i) Given On Z^+, defined * by a * b = a – b
If a = 1 and b = 2 in Z^+, then
a * b = a – b
= 1 – 2
= -1 ∉ Z^+ [because Z^+ is the set of non-negative integers]
For a = 1 and b = 2,
a * b ∉ Z^+

Thus, * is not a binary operation on Z^+.

(ii) Given Z^+, define * by a*b = a b
Let a, b ∈ Z^+
⇒ a, b ∈ Z^+
⇒ a * b ∈ Z^+
Thus, * is a binary operation on R.

(iii) Given on R, define by a*b = ab^2
Let a, b ∈ R
⇒ a, b^2 ∈ R
⇒ ab^2 ∈ R
⇒ a * b ∈ R
Thus, * is a binary operation on R.

(iv) Given on Z^+ define * by a * b = |a − b|
Let a, b ∈ Z^+
⇒ | a – b | ∈ Z^+
⇒ a * b ∈ Z^+
Therefore,
a * b ∈ Z^+, ∀ a, b ∈ Z^+
Thus, * is a binary operation on Z^+.

(v) Given on Z^+ define * by a * b = a
Let a, b ∈ Z^+
⇒ a ∈ Z^+
⇒ a * b ∈ Z^+
Therefore, a * b ∈ Z^+ ∀ a, b ∈ Z^+
Thus, * is a binary operation on Z^+.

(vi) Given On R, define * by a * b = a + $4b^2$
Let a, b ∈ R
⇒ a, $4b^2$ ∈ R
⇒ a + $4b^2$ ∈ R
⇒ a * b ∈ R
Therefore, a *b ∈ R, ∀ a, b ∈ R
Thus, * is a binary operation on R.

3. Let * be a binary operation on the set I of integers, defined by a * b = 2a + b − 3. Find the value of 3 * 4.

Solution:
Given $a * b = 2a + b - 3$
3 * 4 = 2 (3) + 4 – 3
= 6 + 4 – 3
= 7

4. Is * defined on the set {1, 2, 3, 4, 5} by a * b = LCM of a and b a binary operation? Justify your answer.

Solution:

LCM	1	2	3	4	5
1	1	2	3	4	5
2	2	2	6	4	10
3	3	5	3	12	15
4	4	4	12	4	20
5	5	10	15	20	5

In the given composition table, all the elements are not in the set {1, 2, 3, 4, 5}.
If we consider a = 2 and b = 3, a * b = LCM of a and b = 6 ∉ {1, 2, 3, 4, 5}.
Thus, * is not a binary operation on {1, 2, 3, 4, 5}.

5. Let S = {a, b, c}. Find the total number of binary operations on *S*.

Solution:
Number of binary operations on a set with n elements is n^{n^2}
Here, S = {a, b, c}
Number of elements in S = 3
Number of binary operations on a set with 3 elements is 3^{3^2}

EXERCISE 3.2 PAGE NO: 3.12

1. Let '*' be a binary operation on N defined by a * b = l.c.m. (a, b) for all a, b ∈ N
(i) Find 2 * 4, 3 * 5, 1 * 6.
(ii) Check the commutativity and associativity of '*' on N.

Solution:
(i) Given a * b = 1.c.m. (a, b)
2 * 4 = l.c.m. (2, 4)
= 4
3 * 5 = l.c.m. (3, 5)
= 15
1 * 6 = l.c.m. (1, 6)
= 6

(ii) We have to prove commutativity of *
Let a, b ∈ N
a * b = l.c.m (a, b)
= l.c.m (b, a)
= b * a
Therefore
a * b = b * a ∀ a, b ∈ N
Thus * is commutative on N.
Now we have to prove associativity of *
Let a, b, c ∈ N
a * (b * c) = a * l.c.m. (b, c)
= l.c.m. (a, (b, c))
= l.c.m (a, b, c)
(a * b) * c = l.c.m. (a, b) * c
= l.c.m. ((a, b), c)
= l.c.m. (a, b, c)
Therefore
(a * (b * c) = (a * b) * c, ∀ a, b , c ∈ N
Thus, * is associative on N.

2. Determine which of the following binary operation is associative and which is

commutative:
(i) * on N defined by a * b = 1 for all a, b ∈ N
(ii) * on Q defined by a * b = (a + b)/2 for all a, b ∈ Q

Solution:
(i) We have to prove commutativity of *
Let a, b ∈ N
a * b = 1
b * a = 1
Therefore,
a * b = b * a, for all a, b ∈ N
Thus * is commutative on N.
Now we have to prove associativity of *
Let a, b, c ∈ N
Then a * (b * c) = a * (1)
= 1
(a * b) *c = (1) * c
= 1
Therefore a * (b * c) = (a * b) *c for all a, b, c ∈ N
Thus, * is associative on N.

(ii) First we have to prove commutativity of *
Let a, b ∈ N
a * b = (a + b)/2
= (b + a)/2
= b * a
Therefore, a * b = b * a, ∀ a, b ∈ N
Thus * is commutative on N.
Now we have to prove associativity of *
Let a, b, c ∈ N
a * (b * c) = a * (b + c)/2
= [a + (b + c)]/2
= (2a + b + c)/4
Now, (a * b) * c = (a + b)/2 * c
= [(a + b)/2 + c] /2
= (a + b + 2c)/4
Thus, a * (b * c) ≠ (a * b) * c

If a = 1, b= 2, c = 3
1 * (2 * 3) = 1 * (2 + 3)/2
= 1 * (5/2)
= [1 + (5/2)]/2
= 7/4
(1 * 2) * 3 = (1 + 2)/2 * 3
= 3/2 * 3
= [(3/2) + 3]/2
= 4/9
Therefore, there exist a = 1, b = 2, c = 3 ∈ N such that a * (b * c) ≠ (a * b) * c
Thus, * is not associative on N.

3. Let A be any set containing more than one element. Let '*' be a binary operation on A defined by a * b = b for all a, b ∈ A Is '*' commutative or associative on A?

Solution:
Let a, b ∈ A
Then, a * b = b
b * a = a
Therefore a * b ≠ b * a
Thus, * is not commutative on A
Now we have to check associativity:
Let a, b, c ∈ A
a * (b * c) = a * c
= c
Therefore
a * (b * c) = (a * b) * c, ∀ a, b, c ∈ A
Thus, * is associative on A

4. Check the commutativity and associativity of each of the following binary operations:
(i) '*' on Z defined by a * b = a + b + a b for all a, b ∈ Z
(ii) '*' on N defined by a * b = 2^{ab} for all a, b ∈ N
(iii) '*' on Q defined by a * b = a – b for all a, b ∈ Q
(iv) '⊙' on Q defined by a ⊙ b = $a^2 + b^2$ for all a, b ∈ Q
(v) 'o' on Q defined by a o b = (ab/2) for all a, b ∈ Q
(vi) '*' on Q defined by a * b = ab^2 for all a, b ∈ Q

(vii) '*' on Q defined by a * b = a + a b for all a, b ∈ Q
(viii) '*' on R defined by a * b = a + b -7 for all a, b ∈ R
(ix) '*' on Q defined by a * b = $(a - b)^2$ for all a, b ∈ Q
(x) '*' on Q defined by a * b = a b + 1 for all a, b ∈ Q
(xi) '*' on N defined by a * b = a^b for all a, b ∈ N
(xii) '*' on Z defined by a * b = a – b for all a, b ∈ Z
(xiii) '*' on Q defined by a * b = (ab/4) for all a, b ∈ Q
(xiv) '*' on Z defined by a * b = a + b – ab for all a, b ∈ Z
(xv) '*' on Q defined by a * b = gcd (a, b) for all a, b ∈ Q

Solution:
(i) First we have to check commutativity of *
Let a, b ∈ Z
Then a * b = a + b + ab
= b + a + ba
= b * a
Therefore,
a * b = b * a, ∀ a, b ∈ Z
Now we have to prove associativity of *
Let a, b, c ∈ Z, Then,
a * (b * c) = a * (b + c + b c)
= a + (b + c + b c) + a (b + c + b c)
= a + b + c + b c + a b + a c + a b c
(a * b) * c = (a + b + a b) * c
= a + b + a b + c + (a + b + a b) c
= a + b + a b + c + a c + b c + a b c
Therefore,
a * (b * c) = (a * b) * c, ∀ a, b, c ∈ Z
Thus, * is associative on Z.

(ii) First we have to check commutativity of *
Let a, b ∈ N
a * b = 2^{ab}
= 2^{ba}
= b * a
Therefore, a * b = b * a, ∀ a, b ∈ N
Thus, * is commutative on N

Now we have to check associativity of *
Let a, b, c ∈ N
Then, a * (b * c) = a * (2^{bc})
= $2^{a*2^{bc}}$
(a * b) * c = (2^{ab}) * c
= $2^{ab*2^{c}}$
Therefore, a * (b * c) ≠ (a * b) * c
Thus, * is not associative on N

(iii) First we have to check commutativity of *
Let a, b ∈ Q, then
a * b = a – b
b * a = b – a
Therefore, a * b ≠ b * a
Thus, * is not commutative on Q
Now we have to check associativity of *
Let a, b, c ∈ Q, then
a * (b * c) = a * (b – c)
= a – (b – c)
= a – b + c
(a * b) * c = (a – b) * c
= a – b – c
Therefore,
a * (b * c) ≠ (a * b) * c
Thus, * is not associative on Q

(iv) First we have to check commutativity of ⊙
Let a, b ∈ Q, then
a ⊙ b = $a^2 + b^2$
= $b^2 + a^2$
= b ⊙ a
Therefore, a ⊙ b = b ⊙ a, ∀ a, b ∈ Q
Thus, ⊙ on Q
Now we have to check associativity of ⊙
Let a, b, c ∈ Q, then
a ⊙ (b ⊙ c) = a ⊙ ($b^2 + c^2$)
= $a^2 + (b^2 + c^2)^2$

$= a^2 + b^4 + c^4 + 2b^2c^2$

$(a \odot b) \odot c = (a^2 + b^2) \odot c$

$= (a^2 + b^2)^2 + c^2$

$= a^4 + b^4 + 2a^2b^2 + c^2$

Therefore,

$(a \odot b) \odot c \neq a \odot (b \odot c)$

Thus, $\odot$ is not associative on Q.

(v) First we have to check commutativity of o

Let a, b ∈ Q, then

a o b = (ab/2)

= (b a/2)

= b o a

Therefore, a o b = b o a, ∀ a, b ∈ Q

Thus, o is commutative on Q

Now we have to check associativity of o

Let a, b, c ∈ Q, then

a o (b o c) = a o (b c/2)

= [a (b c/2)]/2

= [a (b c/2)]/2

= (a b c)/4

(a o b) o c = (ab/2) o c

= [(ab/2) c] /2

= (a b c)/4

Therefore a o (b o c) = (a o b) o c, ∀ a, b, c ∈ Q

Thus, o is associative on Q.

(vi) First we have to check commutativity of *

Let a, b ∈ Q, then

$a * b = ab^2$

$b * a = ba^2$

Therefore,

$a * b \neq b * a$

Thus, * is not commutative on Q

Now we have to check associativity of *

Let a, b, c ∈ Q, then

$a * (b * c) = a * (bc^2)$

$= a (bc^2)^2$
$= ab^2 c^4$
$(a * b) * c = (ab^2) * c$
$= ab^2c^2$
Therefore $a * (b * c) \neq (a * b) * c$
Thus, * is not associative on Q.

(vii) First we have to check commutativity of *
Let a, b ∈ Q, then
$a * b = a + ab$
$b * a = b + ba$
$= b + ab$
Therefore, $a * b \neq b * a$
Thus, * is not commutative on Q.
Now we have to prove associativity on Q.
Let a, b, c ∈ Q, then
$a * (b * c) = a * (b + b c)$
$= a + a (b + b c)$
$= a + ab + a b c$
$(a * b) * c = (a + a b) * c$
$= (a + a b) + (a + a b) c$
$= a + a b + a c + a b c$
Therefore $a * (b * c) \neq (a * b) * c$
Thus, * is not associative on Q.

(viii) First we have to check commutativity of *
Let a, b ∈ R, then
$a * b = a + b – 7$
$= b + a – 7$
$= b * a$
Therefore,
$a * b = b * a$, for all a, b ∈ R
Thus, * is commutative on R
Now we have to prove associativity of * on R.
Let a, b, c ∈ R, then
$a * (b * c) = a * (b + c – 7)$
$= a + b + c -7 -7$

= a + b + c – 14
(a * b) * c = (a + b – 7) * c
= a + b – 7 + c - 7
= a + b + c – 14
Therefore,
a * (b * c) = (a * b) * c, for all a, b, c ∈ R
Thus, * is associative on R.

(ix) First we have to check commutativity of *
Let a, b ∈ Q, then
a * b = $(a – b)^2$
= $(b – a)^2$
= b * a
Therefore,
a * b = b * a, for all a, b ∈ Q
Thus, * is commutative on Q
Now we have to prove associativity of * on Q
Let a, b, c ∈ Q, then
a * (b * c) = a * $(b – c)^2$
= a * $(b^2 + c^2 – 2\,b\,c)$
= $(a – b^2 – c^2 + 2bc)^2$
(a * b) * c = $(a – b)^2$ * c
= $(a^2 + b^2 – 2ab)$ * c
= $(a^2 + b^2 – 2ab – c)^2$
Therefore, a * (b * c) ≠ (a * b) * c
Thus, * is not associative on Q.

(x) First we have to check commutativity of *
Let a, b ∈ Q, then
a * b = ab + 1
= ba + 1
= b * a
Therefore
a * b = b * a, for all a, b ∈ Q
Thus, * is commutative on Q
Now we have to prove associativity of * on Q
Let a, b, c ∈ Q, then

a * (b * c) = a * (bc + 1)
= a (b c + 1) + 1
= a b c + a + 1
(a * b) * c = (ab + 1) * c
= (ab + 1) c + 1
= a b c + c + 1
Therefore, a * (b * c) ≠ (a * b) * c
Thus, * is not associative on Q.

(xi) First we have to check commutativity of *
Let a, b ∈ N, then
a * b = a^b
b * a = b^a
Therefore, a * b ≠ b * a
Thus, * is not commutative on N.
Now we have to check associativity of *
a * (b * c) = a * (b^c)
= a^{b^c}
(a * b) * c = (a^b) * c
= $(a^b)^c$
= a^{bc}
Therefore, a * (b * c) ≠ (a * b) * c
Thus, * is not associative on N

(xii) First we have to check commutativity of *
Let a, b ∈ Z, then
a * b = a – b
b * a = b – a
Therefore,
a * b ≠ b * a
Thus, * is not commutative on Z.
Now we have to check associativity of *
Let a, b, c ∈ Z, then
a * (b * c) = a * (b – c)
= a – (b – c)
= a – (b + c)
(a * b) * c = (a – b) – c

= a – b – c
Therefore, a * (b * c) ≠ (a * b) * c
Thus, * is not associative on Z

(xiii) First we have to check commutativity of *
Let a, b ∈ Q, then
a * b = (ab/4)
= (ba/4)
= b * a
Therefore, a * b = b * a, for all a, b ∈ Q
Thus, * is commutative on Q
Now we have to check associativity of *
Let a, b, c ∈ Q, then
a * (b * c) = a * (b c/4)
= [a (b c/4)]/4
= (a b c/16)
(a * b) * c = (ab/4) * c
= [(ab/4) c]/4
= a b c/16
Therefore,
a * (b * c) = (a * b) * c for all a, b, c ∈ Q
Thus, * is associative on Q.

(xiv) First we have to check commutativity of *
Let a, b ∈ Z, then
a * b = a + b – ab
= b + a – ba
= b * a
Therefore, a * b = b * a, for all a, b ∈ Z
Thus, * is commutative on Z.
Now we have to check associativity of *
Let a, b, c ∈ Z
a * (b * c) = a * (b + c – b c)
= a + b + c- b c – ab – ac + a b c
(a * b) * c = (a + b – a b) c
= a + b – ab + c – (a + b – ab) c
= a + b + c – ab – ac – bc + a b c

Therefore,
a * (b * c) = (a * b) * c, for all a, b, c ∈ Z
Thus, * is associative on Z.

(xv) First we have to check commutativity of *
Let a, b ∈ N, then
a * b = gcd (a, b)
= gcd (b, a)
= b * a
Therefore, a * b = b * a, for all a, b ∈ N
Thus, * is commutative on N.
Now we have to check associativity of *
Let a, b, c ∈ N
a * (b * c) = a * [gcd (a, b)]
= gcd (a, b, c)
(a * b) * c = [gcd (a, b)] * c
= gcd (a, b, c)
Therefore,
a * (b * c) = (a * b) * c, for all a, b, c ∈ N
Thus, * is associative on N.

5. If the binary operation o is defined by a0b = a + b – ab on the set Q – {-1} of all rational numbers other than 1, show that o is commutative on Q – [1].

Solution:
Let a, b ∈ Q – {-1}.
Then aob = a + b – ab
= b+ a – ba
= boa
Therefore,
aob = boa for all a, b ∈ Q – {-1}
Thus, o is commutative on Q – {-1}

6. Show that the binary operation * on Z defined by a * b = 3a + 7b is not commutative?

Solution:

Let a, b ∈ Z
a * b = 3a + 7b
b * a = 3b + 7a
Thus, a * b ≠ b * a
Let a = 1 and b = 2
1 * 2 = 3 × 1 + 7 × 2
= 3 + 14
= 17
2 * 1 = 3 × 2 + 7 × 1
= 6 + 7
= 13
Therefore, there exist a = 1, b = 2 ∈ Z such that a * b ≠ b * a
Thus, * is not commutative on Z.

7. On the set Z of integers a binary operation * is defined by a 8 b = ab + 1 for all a, b ∈ Z. Prove that * is not associative on Z.

Solution:
Let a, b, c ∈ Z
a * (b * c) = a * (bc + 1)
= a (bc + 1) + 1
= a b c + a + 1
(a * b) * c = (ab+ 1) * c
= (ab + 1) c + 1
= a b c + c + 1
Thus, a * (b * c) ≠ (a * b) * c
Thus, * is not associative on Z.

EXERCISE 3.3 PAGE NO: 3.15

1. Find the identity element in the set I^+ of all positive integers defined by a * b = a + b for all a, b ∈ I^+.

Solution:

Let e be the identity element in I^+ with respect to * such that
a * e = a = e * a, ∀ a ∈ I^+
a * e = a and e * a = a, ∀ a ∈ I^+
a + e = a and e + a = a, ∀ a ∈ I^+
e = 0, ∀ a ∈ I^+
Thus, 0 is the identity element in I^+ with respect to *.

2. Find the identity element in the set of all rational numbers except – 1 with respect to * defined by a * b = a + b + ab

Solution:

Let e be the identity element in I^+ with respect to * such that
a * e = a = e * a, ∀ a ∈ Q – {-1}
a * e = a and e * a = a, ∀ a ∈ Q – {-1}
a + e + ae = a and e + a + ea = a, ∀ a ∈ Q – {-1}
e + ae = 0 and e + ea = 0, ∀ a ∈ Q – {-1}
e (1 + a) = 0 and e (1 + a) = 0, ∀ a ∈ Q – {-1}
e = 0, ∀ a ∈ Q – {-1} [because a not equal to -1]
Thus, 0 is the identity element in Q – {-1} with respect to *.

EXERCISE 3.4 PAGE NO: 3.25

1. Let * be a binary operation on Z defined by a * b = a + b – 4 for all a, b ∈ Z.
(i) Show that * is both commutative and associative.
(ii) Find the identity element in Z
(iii) Find the invertible element in Z.

Solution:
(i) First we have to prove commutativity of *
Let a, b ∈ Z. then,
a * b = a + b – 4
= b + a – 4
= b * a
Therefore,
a * b = b * a, ∀ a, b ∈ Z
Thus, * is commutative on Z.
Now we have to prove associativity of Z.
Let a, b, c ∈ Z. then,
a * (b * c) = a * (b + c - 4)
= a + b + c -4 – 4
= a + b + c – 8
(a * b) * c = (a + b – 4) * c
= a + b – 4 + c – 4
= a + b + c – 8
Therefore,
a * (b * c) = (a * b) * c, for all a, b, c ∈ Z
Thus, * is associative on Z.

(ii) Let e be the identity element in Z with respect to * such that
a * e = a = e * a ∀ a ∈ Z
a * e = a and e * a = a, ∀ a ∈ Z
a + e – 4 = a and e + a – 4 = a, ∀ a ∈ Z
e = 4, ∀ a ∈ Z
Thus, 4 is the identity element in Z with respect to *.

(iii) Let a ∈ Z and b ∈ Z be the inverse of a. Then,

a * b = e = b * a
a * b = e and b * a = e
a + b – 4 = 4 and b + a – 4 = 4
b = 8 – a ∈ Z
Thus, 8 – a is the inverse of a ∈ Z

2. Let * be a binary operation on Q_0 (set of non-zero rational numbers) defined by a * b = (3ab/5) for all a, b ∈ Q_0. Show that * is commutative as well as associative. Also, find its identity element, if it exists.

Solution:
First we have to prove commutativity of *
Let a, b ∈ Q_0
a * b = (3ab/5)
= (3ba/5)
= b * a
Therefore, a * b = b * a, for all a, b ∈ Q_0
Now we have to prove associativity of *
Let a, b, c ∈ Q_0
a * (b * c) = a * (3bc/5)
= [a (3 bc/5)] /5
= 3 abc/25
(a * b) * c = (3 ab/5) * c
= [(3 ab/5) c]/ 5
= 3 abc /25
Therefore a * (b * c) = (a * b) * c, for all a, b, c ∈ Q_0
Thus * is associative on Q_0
Now we have to find the identity element
Let e be the identity element in Z with respect to * such that
a * e = a = e * a ∀ a ∈ Q_0
a * e = a and e * a = a, ∀ a ∈ Q_0
3ae/5 = a and 3ea/5 = a, ∀ a ∈ Q_0
e = 5/3 ∀ a ∈ Q_0 [because a is not equal to 0]
Thus, 5/3 is the identity element in Q_0 with respect to *.

3. Let * be a binary operation on Q – {-1} defined by a * b = a + b + ab for all a, b ∈ Q – {-1}. Then,

(i) Show that * is both commutative and associative on Q – {-1}
(ii) Find the identity element in Q – {-1}
(iii) Show that every element of Q – {-1} is invertible. Also, find inverse of an arbitrary element.

Solution:
(i) First we have to check commutativity of *
Let a, b ∈ Q – {-1}
Then a * b = a + b + ab
= b + a + ba
= b * a
Therefore,
a * b = b * a, ∀ a, b ∈ Q – {-1}
Now we have to prove associativity of *
Let a, b, c ∈ Q – {-1}, Then,
a * (b * c) = a * (b + c + b c)
= a + (b + c + b c) + a (b + c + b c)
= a + b + c + b c + a b + a c + a b c
(a * b) * c = (a + b + a b) * c
= a + b + a b + c + (a + b + a b) c
= a + b + a b + c + a c + b c + a b c
Therefore,
a * (b * c) = (a * b) * c, ∀ a, b, c ∈ Q – {-1}
Thus, * is associative on Q – {-1}.

(ii) Let e be the identity element in I^+ with respect to * such that
a * e = a = e * a, ∀ a ∈ Q – {-1}
a * e = a and e * a = a, ∀ a ∈ Q – {-1}
a + e + ae = a and e + a + ea = a, ∀ a ∈ Q – {-1}
e + ae = 0 and e + ea = 0, ∀ a ∈ Q – {-1}
e (1 + a) = 0 and e (1 + a) = 0, ∀ a ∈ Q – {-1}
e = 0, ∀ a ∈ Q – {-1} [because a not equal to -1]
Thus, 0 is the identity element in Q – {-1} with respect to *.

(iii) Let a ∈ Q – {-1} and b ∈ Q – {-1} be the inverse of a. Then,
a * b = e = b * a
a * b = e and b * a = e

a + b + ab = 0 and b + a + ba = 0
b (1 + a) = - a Q – {-1}
b = -a/1 + a Q – {-1} [because a not equal to -1]
Thus, -a/1 + a is the inverse of a ∈ Q – {-1}

4. Let A = R_0 × R, where R_0 denote the set of all non-zero real numbers. A binary operation 'O' is defined on A as follows: (a, b) O (c, d) = (ac, bc + d) for all (a, b), (c, d) ∈ R_0 × R.
(i) Show that 'O' is commutative and associative on A
(ii) Find the identity element in A
(iii) Find the invertible element in A.

Solution:
(i) Let X = (a, b) and Y = (c, d) ∈ A, ∀ a, c ∈ R_0 and b, d ∈ R
Then, X O Y = (ac, bc + d)
And Y O X = (ca, da + b)
Therefore,
X O Y = Y O X, ∀ X, Y ∈ A
Thus, O commutative on A.
Now we have to check associativity of O
Let X = (a, b), Y = (c, d) and Z = (e, f), ∀ a, c, e ∈ R_0 and b, d, f ∈ R
X O (Y O Z) = (a, b) O (ce, de + f)
= (ace, bce + de + f)
(X O Y) O Z = (ac, bc + d) O (e, f)
= (ace, (bc + d) e + f)
= (ace, bce + de + f)
Therefore, X O (Y O Z) = (X O Y) O Z, ∀ X, Y, Z ∈ A

(ii) Let E = (x, y) be the identity element in A with respect to O, ∀ x ∈ R_0 and y ∈ R
Such that,
X O E = X = E O X, ∀ X ∈ A
X O E = X and EOX = X
(ax, bx +y) = (a, b) and (xa, ya + b) = (a, b)
Considering (ax, bx + y) = (a, b)
ax = a
x = 1
And bx + y = b

y = 0 [since x = 1]
Considering (xa, ya + b) = (a, b)
xa = a
x = 1
And ya + b = b
y = 0 [since x = 1]
Therefore (1, 0) is the identity element in A with respect to O.

(iii) Let F = (m, n) be the inverse in A ∀ m ∈ R_0 and n ∈ R
X O F = E and F O X = E
(am, bm + n) = (1, 0) and (ma, na + b) = (1, 0)
Considering (am, bm + n) = (1, 0)
am = 1
m = 1/a
And bm + n = 0
n = -b/a [since m = 1/a]
Considering (ma, na + b) = (1, 0)
ma = 1
m = 1/a
And na + b = 0
n = -b/a
Therefore the inverse of (a, b) ∈ A with respect to O is (1/a, -1/a)

EXERCISE 3.5 PAGE NO: 3.33

1. Construct the composition table for $\times_4$ on set S = {0, 1, 2, 3}.

Solution:

Given that $\times_4$ on set S = {0, 1, 2, 3}

Here,

$1 \times_4 1$ = remainder obtained by dividing 1 × 1 by 4

= 1

$0 \times_4 1$ = remainder obtained by dividing 0 × 1 by 4

= 0

$2 \times_4 3$ = remainder obtained by dividing 2 × 3 by 4

= 2

$3 \times_4 3$ = remainder obtained by dividing 3 × 3 by 4

= 1

So, the composition table is as follows:

$\times_4$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	2	2
3	0	3	2	1

2. Construct the composition table for $+_5$ on set S = {0, 1, 2, 3, 4}

Solution:

$1 +_5 1$ = remainder obtained by dividing 1 + 1 by 5

= 2

$3 +_5 1$ = remainder obtained by dividing 3 + 1 by 5

= 2

$4 +_5 1$ = remainder obtained by dividing 4 + 1 by 5

= 3

So, the composition table is as follows:

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

3. Construct the composition table for $\times_6$ on set S = {0, 1, 2, 3, 4, 5}.

Solution:

Here,

$1 \times_6 1$ = remainder obtained by dividing 1×1 by 6

= 1

$3 \times_6 4$ = remainder obtained by dividing 3×4 by 6

= 0

$4 \times_6 5$ = remainder obtained by dividing 4×5 by 6

= 2

So, the composition table is as follows:

$\times_6$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

4. Construct the composition table for $\times_5$ on set Z_5 = {0, 1, 2, 3, 4}

Solution:
Here,
1 $\times_5$ 1 = remainder obtained by dividing 1 × 1 by 5
= 1
3 $\times_5$ 4 = remainder obtained by dividing 3 × 4 by 5
= 2
4 $\times_5$ 4 = remainder obtained by dividing 4 × 4 by 5
= 1
So, the composition table is as follows:

$\times_5$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

5. For the binary operation $\times_{10}$ set S = {1, 3, 7, 9}, find the inverse of 3.

Solution:
Here,
1 $\times_{10}$ 1 = remainder obtained by dividing 1 × 1 by 10
= 1
3 $\times_{10}$ 7 = remainder obtained by dividing 3 × 7 by 10
= 1
7 $\times_{10}$ 9 = remainder obtained by dividing 7 × 9 by 10
= 3
So, the composition table is as follows:

$\times_{10}$	1	3	7	9
1	1	3	7	9
3	3	9	1	7
7	7	1	9	3
9	9	7	3	1

From the table we can observe that elements of first row as same as the top-most row.
So, $1 \in S$ is the identity element with respect to $\times_{10}$
Now we have to find inverse of 3
$3 \times_{10} 7 = 1$
So the inverse of 3 is 7.