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1. Discuss the applicability of Rolle's Theorem for the following functions on the indicated intervals:

(i) $f(x) = 3 + (x-2)^{\frac{2}{3}}$ on $[1, 3]$

Solution:

Given function is

$\Rightarrow f(x) = 3 + (x-2)^{\frac{2}{3}}$ on [1, 3]

Let us check the differentiability of the function f(x).

Now we have to find the derivative of f(x),

$\Rightarrow f'(x) = \frac{d}{dx}\left(3 + (x-2)^{\frac{2}{3}}\right)$

$\Rightarrow f'(x) = \frac{d(3)}{dx} + \frac{d\left((x-2)^{\frac{2}{3}}\right)}{dx}$

$\Rightarrow f'(x) = 0 + \frac{2}{3}(x-2)^{\frac{2}{3}-1}$

$\Rightarrow f'(x) = \frac{2}{3}(x-2)^{-\frac{1}{3}}$

$\Rightarrow f'(x) = \frac{2}{3(x-2)^{\frac{1}{3}}}$

Now we have to check differentiability at the value of x = 2

$\Rightarrow \lim_{x\to 2} f'(x) = \lim_{x\to 2} \frac{2}{3(x-2)^{\frac{1}{3}}}$

$\Rightarrow \lim_{x\to 2} f'(x) = \frac{2}{3(2-2)^{\frac{1}{3}}}$

$\Rightarrow \lim_{x\to 2} f'(x) = \frac{2}{3(0)}$

$\Rightarrow \lim_{x \to 2} f'(x) = \text{undefined}$

∴ f is not differentiable at x = 2, so it is not differentiable in the closed interval (1, 3).

So, Rolle's theorem is not applicable for the function f on the interval [1, 3].

(ii) f (x) = [x] for -1 < x ≤ 1, where [x] denotes the greatest integer not exceeding x

Solution:

Given function is f (x) = [x], – 1≤x≤1 where [x] denotes the greatest integer not exceeding x.

Let us check the continuity of the function f.

Here in the interval x ϵ [– 1, 1], the function has to be Right continuous at x = 1 and left continuous at x = 1.

$\Rightarrow \lim_{x \to 1+} f(x) = \lim_{x \to 1+} [x]$

$\Rightarrow \lim_{x \to 1+} f(x) = \lim_{x \to 1+h} [x]$ Where h>0.

$\Rightarrow \lim_{x \to 1+} f(x) = \lim_{h \to 0} 1$

$\Rightarrow \lim_{x \to 1+} f(x) = 1$ ...... (1)

$\Rightarrow \lim_{x \to 1-} f(x) = \lim_{x \to 1-} [x]$

$\Rightarrow \lim_{x \to 1-} f(x) = \lim_{x \to 1-h} [x]$, where h>0

$\Rightarrow \lim_{x \to 1-} f(x) = \lim_{h \to 0} 0$

$\Rightarrow \lim_{x \to 1-} f(x) = 0$ ...... (2)

From (1) and (2), we can see that the limits are not the same so, the function is not continuous in the interval [– 1, 1].

∴ Rolle's Theorem is not applicable for the function f in the interval [– 1, 1].

(iii) $f(x) = \sin\frac{1}{x}$ **for** $-1 \le x \le 1$

Solution:

Given function is $f(x) = \sin\left(\frac{1}{x}\right)$ for $-1 \le x \le 1$

Let us check the continuity of the function 'f' at the value of x = 0.We cannot directly find the value of limit at x = 0, as the function is not valid at x = 0. So, we take the limit on either sides or x = 0, and we check whether they are equal or not.

So consider RHL:

$$\Rightarrow \lim_{x\to0+} f(x) = \lim_{x\to0+} \sin\left(\frac{1}{x}\right)$$

We assume that the limit $\lim_{h\to0} \sin\left(\frac{1}{h}\right) = k$, $k \in [-1, 1]$.

$$\Rightarrow \lim_{x\to0+} f(x) = \lim_{x\to0+h} \sin\left(\frac{1}{x}\right)\text{, where h>0}$$

$$\Rightarrow \lim_{x\to0+} f(x) = \lim_{h\to0} \sin\left(\frac{1}{h+0}\right)$$

$$\Rightarrow \lim_{x\to0+} f(x) = \lim_{h\to0} \sin\left(\frac{1}{h}\right)$$

$$\Rightarrow \lim_{x\to0+} f(x) = k \quad \text{...... (1)}$$

Now consider LHL:

$$\Rightarrow \lim_{x\to0-} f(x) = \lim_{x\to0-} \sin\left(\frac{1}{x}\right)$$

$$\Rightarrow \lim_{x\to0-} f(x) = \lim_{x\to0-h} \sin\left(\frac{1}{x}\right)\text{, where h>0}$$

$$\Rightarrow \lim_{x\to0-} f(x) = \lim_{h\to0} \sin\left(\frac{1}{0-h}\right)$$

$$\Rightarrow \lim_{x\to0-} f(x) = \lim_{h\to0} \sin\left(\frac{1}{-h}\right)$$

$\Rightarrow \lim_{x\to 0-} f(x) = \lim_{h\to 0} -\sin\left(\frac{1}{h}\right)$

$\Rightarrow \lim_{x\to 0-} f(x) = -\lim_{h\to 0} \sin\left(\frac{1}{h}\right)$

$\Rightarrow \lim_{x\to 0-} f(x) = -k$ ...... (2)

From (1) and (2), we can see that the Right hand and left – hand limits are not equal, so the function 'f' is not continuous at x = 0.

∴ Rolle's Theorem is not applicable to the function 'f' in the interval [– 1, 1].

(iv) f (x) = $2x^2 - 5x + 3$ on [1, 3]

Solution:

Given function is f (x) = $2x^2 - 5x + 3$ on [1, 3]
Since given function f is a polynomial. So, it is continuous and differentiable everywhere.
Now, we find the values of function at the extreme values.
⇒ f (1) = $2(1)^2 – 5(1) + 3$
⇒ f (1) = 2 – 5 + 3
⇒ f (1) = 0...... (1)
⇒ f (3) = $2(3)^2 – 5(3) + 3$
⇒ f (3) = 2(9)–15 + 3
⇒ f (3) = 18 – 12
⇒ f (3) = 6...... (2)
From (1) and (2), we can say that, f (1) ≠ f (3)
∴ Rolle's Theorem is not applicable for the function f in interval [1, 3].

(v) f (x) = $x^{2/3}$ on [-1, 1]

Solution:

Given function is $f(x) = x^{\frac{2}{3}}$ on [– 1, 1]

Now we have to find the derivative of the given function:

$\Rightarrow f'(x) = \frac{d\left(x^{\frac{2}{3}}\right)}{dx}$

$\Rightarrow f'(x) = \frac{2}{3}x^{\frac{2}{3}-1}$

$\Rightarrow f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$

$\Rightarrow f'(x) = \frac{2}{3x^{\frac{1}{3}}}$

Now we have to check the differentiability of the function at x = 0.

$\Rightarrow \lim_{x\to 0} f'(x) = \lim_{x\to 0} \frac{2}{3x^{\frac{1}{3}}}$

$\Rightarrow \lim_{x\to 0} f'(x) = \frac{2}{3(0)^{\frac{1}{3}}}$

$\Rightarrow \lim_{x\to 0} f'(x) = \text{undefined}$

Since the limit for the derivative is undefined at x = 0, we can say that f is not differentiable at x = 0.

∴ Rolle's Theorem is not applicable to the function 'f' on [– 1, 1].

(vi) $f(x) = \begin{cases} -4x + 5, & 0 \le x \le 1 \\ 2x - 3, & 1 < x \le 2 \end{cases}$

Solution:

Given function is $f(x) = \begin{cases} -4x + 5, 0 \le x \le 1 \\ 2x - 3, 1 < x \le 2 \end{cases}$

Now we have to check the continuity at x = 1 as the equation of function changes.

Consider LHL:

$\Rightarrow \lim_{x\to 1-} f(x) = \lim_{x\to 1-} -4x + 5$

$\Rightarrow \lim_{x\to 1-} f(x) = -4(1) + 5$

$\Rightarrow \lim_{x \to 1-} f(x) = 1$ ...... (1)

Now consider RHL:

$\Rightarrow \lim_{x \to 1+} f(x) = \lim_{x \to 1+} 2x - 3$

$\Rightarrow \lim_{x \to 1+} f(x) = 2(0) - 3$

$\Rightarrow \lim_{x \to 1+} f(x) = -1$ ...... (2)

From (1) and (2), we can see that the values of both side limits are not equal.
So, the function 'f' is not continuous at x = 1.

∴ Rolle's Theorem is not applicable to the function 'f' in the interval [0, 2].

2. Verify the Rolle's Theorem for each of the following functions on the indicated intervals:
(i) f (x) = x^2 – 8x + 12 on [2, 6]

Solution:
Given function is f (x) = x^2 – 8x + 12 on [2, 6]
Since, given function f is a polynomial it is continuous and differentiable everywhere i.e., on R.
Let us find the values at extremes:
⇒ f (2) = 2^2 – 8(2) + 12
⇒ f (2) = 4 – 16 + 12
⇒ f (2) = 0
⇒ f (6) = 6^2 – 8(6) + 12
⇒ f (6) = 36 – 48 + 12
⇒ f (6) = 0
∴ f (2) = f(6), Rolle's theorem applicable for function f on [2,6].
Now we have to find the derivative of f(x)

$\Rightarrow f'(x) = \frac{d(x^2 - 8x + 12)}{dx}$

$\Rightarrow f'(x) = \frac{d(x^2)}{dx} - \frac{d(8x)}{dx} + \frac{d(12)}{dx}$

$\Rightarrow f'(x) = 2x - 8 + 0$

⇒ f′(x) = 2x – 8

We have f′(c) = 0 ϵ [2, 6], from the above definition

⇒ f′(c) = 0

⇒ 2c – 8 = 0

⇒ 2c = 8

$\Rightarrow c = \frac{8}{2}$

⇒ C = 4 ϵ [2, 6]

∴ Rolle's Theorem is verified.

(ii) f(x) = x^2 – 4x + 3 on [1, 3]

Solution:

Given function is f (x) = x^2 – 4x + 3 on [1, 3]
Since, given function f is a polynomial it is continuous and differentiable everywhere i.e., on R. Let us find the values at extremes:
⇒ f (1) = 1^2 – 4(1) + 3
⇒ f (1) = 1 – 4 + 3
⇒ f (1) = 0
⇒ f (3) = 3^2 – 4(3) + 3
⇒ f (3) = 9 – 12 + 3
⇒ f (3) = 0
∴ f (1) = f(3), Rolle's theorem applicable for function 'f' on [1,3].
Let's find the derivative of f(x)

$\Rightarrow f'(x) = \frac{d(x^2 - 4x + 3)}{dx}$

$\Rightarrow f'(x) = \frac{d(x^2)}{dx} - \frac{d(4x)}{dx} + \frac{d(3)}{dx}$

$\Rightarrow f'(x) = 2x - 4 + 0$

⇒ f′(x) = 2x – 4

We have f'(c) = 0, c ϵ (1, 3), from the definition of Rolle's Theorem.

⇒ f'(c) = 0

⇒ 2c – 4 = 0

⇒ 2c = 4

⇒ c = 4/2

⇒ C = 2 ϵ (1, 3)

∴ Rolle's Theorem is verified.

(iii) f (x) = (x – 1) (x – 2)^2 on [1, 2]

Solution:

Given function is f (x) = (x – 1) (x – 2)^2 on [1, 2]

Since, given function f is a polynomial it is continuous and differentiable everywhere that is on R.

Let us find the values at extremes:

⇒ f (1) = (1 – 1) (1 – 2)^2

⇒ f (1) = 0(1)^2

⇒ f (1) = 0

⇒ f (2) = (2 – 1)(2 – 2)^2

⇒ f (2) = 0^2

⇒ f (2) = 0

∴ f (1) = f (2), Rolle's Theorem applicable for function 'f' on [1, 2].

Let's find the derivative of f(x)

$\Rightarrow f'(x) = \frac{d((x-1)(x-2)^2)}{dx}$

Differentiating by using product rule, we get

$\Rightarrow f'(x) = (x-2)^2 \times \frac{d(x-1)}{dx} + (x-1) \times \frac{d((x-2)^2)}{dx}$

⇒ f'(x) = ((x – 2)^2×1) + ((x – 1) × 2 × (x – 2))

⇒ f'(x) = x^2 – 4x + 4 + 2(x^2 – 3x + 2)

⇒ f'(x) = 3x^2 – 10x + 8

We have f'(c) = 0 c ϵ (1, 2), from the definition of Rolle's Theorem.

⇒ f'(c) = 0

$\Rightarrow 3c^2 - 10c + 8 = 0$

$\Rightarrow c = \frac{10 \pm \sqrt{(-10)^2 - (4\times3\times8)}}{2\times3}$

$\Rightarrow c = \frac{10 \pm \sqrt{100-96}}{6}$

$\Rightarrow c = \frac{10 \pm 2}{6}$

$\Rightarrow c = \frac{12}{6}$ or $c = \frac{8}{6}$

$\Rightarrow c = \frac{4}{3} \in (1, 2)$ (neglecting the value 2)

∴ Rolle's Theorem is verified.

(iv) f (x) = x (x – 1)^2 on [0, 1]

Solution:
Given function is $f(x) = x(x-1)^2$ on [0, 1]
Since, given function f is a polynomial it is continuous and differentiable everywhere that is, on R.
Let us find the values at extremes
$\Rightarrow f(0) = 0(0-1)^2$
$\Rightarrow f(0) = 0$
$\Rightarrow f(1) = 1(1-1)^2$
$\Rightarrow f(1) = 0^2$
$\Rightarrow f(1) = 0$
∴ f (0) = f (1), Rolle's theorem applicable for function 'f' on [0,1].
Let's find the derivative of f(x)

$\Rightarrow f'(x) = \frac{d(x(x-1)^2)}{dx}$

Differentiating using product rule:

$\Rightarrow f'(x) = (x-1)^2 \times \frac{d(x)}{dx} + x\frac{d((x-1)^2)}{dx}$

$\Rightarrow f'(x) = ((x-1)^2\times1) + (x\times2\times(x-1))$

⇒ $f'(x) = (x - 1)^2 + 2(x^2 - x)$

⇒ $f'(x) = x^2 - 2x + 1 + 2x^2 - 2x$

⇒ $f'(x) = 3x^2 - 4x + 1$

We have f'(c) = 0 c ϵ (0, 1), from the definition given above.

⇒ $f'(c) = 0$

⇒ $3c^2 - 4c + 1 = 0$

⇒ $c = \frac{4 \pm \sqrt{(-4)^2 - (4 \times 3 \times 1)}}{2 \times 3}$

⇒ $c = \frac{4 \pm \sqrt{16 - 12}}{6}$

⇒ $c = \frac{4 \pm \sqrt{4}}{6}$

⇒ $c = \frac{6}{6}$ or $c = \frac{2}{6}$

⇒ $c = \frac{1}{3} \in (0, 1)$

∴ Rolle's Theorem is verified.

(v) f (x) = $(x^2 - 1)(x - 2)$ on [-1, 2]

Solution:
Given function is f (x) = $(x^2 - 1)(x - 2)$ on [– 1, 2]
Since, given function f is a polynomial it is continuous and differentiable everywhere that is on R.
Let us find the values at extremes:
⇒ $f(-1) = ((-1)^2 - 1)(-1 - 2)$
⇒ $f(-1) = (1 - 1)(-3)$
⇒ $f(-1) = (0)(-3)$
⇒ $f(-1) = 0$
⇒ $f(2) = (2^2 - 1)(2 - 2)$
⇒ $f(2) = (4 - 1)(0)$
⇒ $f(2) = 0$

∴ f (– 1) = f (2), Rolle's theorem applicable for function f on [– 1,2].
Let's find the derivative of f(x)

$\Rightarrow f'(x) = \frac{d((x^2-1)(x-2))}{dx}$

Differentiating using product rule,

$\Rightarrow f'(x) = (x-2) \times \frac{d(x^2-1)}{dx} + (x^2-1)\frac{d(x-2)}{dx}$

$\Rightarrow f'(x) = ((x-2) \times 2x) + ((x^2-1) \times 1)$

$\Rightarrow f'(x) = 2x^2 - 4x + x^2 - 1$

$\Rightarrow f'(x) = 2x^2 - 4x - 1$

We have f'(c) = 0 c ϵ (– 1, 2), from the definition of Rolle's Theorem.

$\Rightarrow f'(c) = 0$

$\Rightarrow 2c^2 - 4c - 1 = 0$

$\Rightarrow c = \frac{4 \pm \sqrt{(-4)^2-(4\times2\times-1)}}{2\times2}$

$\Rightarrow c = \frac{4 \pm \sqrt{16+8}}{4}$

$\Rightarrow c = \frac{4 \pm \sqrt{24}}{4}$

$\Rightarrow c = \frac{4+2\sqrt{6}}{4}$ or $c = \frac{4-2\sqrt{6}}{4}$

$\Rightarrow c = 1 + \frac{\sqrt{6}}{2}$ or $c = 1 - \frac{\sqrt{6}}{2}$

$\Rightarrow c = 1 - \frac{\sqrt{6}}{2} \in (-1, 2)$

∴ Rolle's Theorem is verified.

(vi) f (x) = x (x – 4)^2 on [0, 4]

Solution:

Given function is $f(x) = x(x-4)^2$ on [0, 4]
Since, given function f is a polynomial it is continuous and differentiable everywhere i.e., on R.
Let us find the values at extremes:
$\Rightarrow f(0) = 0(0-4)^2$
$\Rightarrow f(0) = 0$
$\Rightarrow f(4) = 4(4-4)^2$
$\Rightarrow f(4) = 4(0)^2$
$\Rightarrow f(4) = 0$
∴ f (0) = f (4), Rolle's theorem applicable for function 'f' on [0,4].
Let's find the derivative of f(x):

$$\Rightarrow f'(x) = \frac{d(x(x-4)^2)}{dx}$$

Differentiating using product rule

$$\Rightarrow f'(x) = (x-4)^2 \times \frac{d(x)}{dx} + x\frac{d((x-4)^2)}{dx}$$

$\Rightarrow f'(x) = ((x-4)^2 \times 1) + (x \times 2 \times (x-4))$

$\Rightarrow f'(x) = (x-4)^2 + 2(x^2 - 4x)$

$\Rightarrow f'(x) = x^2 - 8x + 16 + 2x^2 - 8x$

$\Rightarrow f'(x) = 3x^2 - 16x + 16$

We have f'(c) = 0 c ϵ (0, 4), from the definition of Rolle's Theorem.

$\Rightarrow f'(c) = 0$

$\Rightarrow 3c^2 - 16c + 16 = 0$

$$\Rightarrow c = \frac{16 \pm \sqrt{(-16)^2 - (4 \times 3 \times 16)}}{2 \times 3}$$

$$\Rightarrow c = \frac{16 \pm \sqrt{256 - 192}}{6}$$

$$\Rightarrow c = \frac{16 \pm \sqrt{64}}{6}$$

$\Rightarrow c = \frac{8}{6}$ or $c = \frac{24}{6}$

$\Rightarrow c = \frac{8}{6} \in (0, 4)$

∴ Rolle's Theorem is verified.

(vii) f (x) = x (x – 2)^2 on [0, 2]

Solution:
Given function is f (x) = x (x – 2)^2 on [0, 2]
Since, given function f is a polynomial it is continuous and differentiable everywhere that is on R.
Let us find the values at extremes:
⇒ f (0) = 0(0 – 2)^2
⇒ f (0) = 0
⇒ f (2) = 2(2 – 2)^2
⇒ f (2) = 2(0)^2
⇒ f (2) = 0
f (0) = f(2), Rolle's theorem applicable for function f on [0,2].
Let's find the derivative of f(x)

$\Rightarrow f'(x) = \frac{d(x(x-2)^2)}{dx}$

Differentiating using UV rule,

$\Rightarrow f'(x) = (x-2)^2 \times \frac{d(x)}{dx} + x\frac{d((x-2)^2)}{dx}$

⇒ f'(x) = ((x – 2)^2×1) + (x×2×(x – 2))

⇒ f'(x) = (x – 2)^2 + 2(x^2 – 2x)

⇒ f'(x) = x^2 – 4x + 4 + 2x^2 – 4x

⇒ f'(x) = 3x^2 – 8x + 4

We have f'(c) = 0 c ϵ (0, 1), from the definition of Rolle's Theorem.

⇒ f'(c) = 0

$\Rightarrow 3c^2 - 8c + 4 = 0$

$\Rightarrow c = \frac{8 \pm \sqrt{(-8)^2 - (4\times3\times4)}}{2\times3}$

$\Rightarrow c = \frac{8\pm\sqrt{64-48}}{6}$

$\Rightarrow c = \frac{8\pm\sqrt{16}}{6}$

$\Rightarrow c = \frac{12}{6}$ or $c = \frac{6}{6}$

$\Rightarrow c = 1 \in (0, 2)$

∴ Rolle's Theorem is verified.

(viii) f (x) = x^2 + 5x + 6 on [-3, -2]

Solution:

Given function is f (x) = $x^2 + 5x + 6$ on [– 3, – 2]

Since, given function f is a polynomial it is continuous and differentiable everywhere i.e., on R. Let us find the values at extremes:

$\Rightarrow f(-3) = (-3)^2 + 5(-3) + 6$

$\Rightarrow f(-3) = 9 - 15 + 6$

$\Rightarrow f(-3) = 0$

$\Rightarrow f(-2) = (-2)^2 + 5(-2) + 6$

$\Rightarrow f(-2) = 4 - 10 + 6$

$\Rightarrow f(-2) = 0$

∴ f (– 3) = f(– 2), Rolle's theorem applicable for function f on [– 3, – 2].

Let's find the derivative of f(x):

$\Rightarrow f'(x) = \frac{d(x^2 + 5x + 6)}{dx}$

$\Rightarrow f'(x) = \frac{d(x^2)}{dx} + \frac{d(5x)}{dx} + \frac{d(6)}{dx}$

$\Rightarrow f'(x) = 2x + 5 + 0$

$\Rightarrow f'(x) = 2x + 5$

We have f'(c) = 0 c ϵ (– 3, – 2), from the definition of Rolle's Theorem

⇒ f'(c) = 0

⇒ 2c + 5 = 0

⇒ 2c = – 5

⇒ $c = -\frac{5}{2}$

⇒ C = – 2.5 ϵ (– 3, – 2)

∴ Rolle's Theorem is verified.

3. Verify the Rolle's Theorem for each of the following functions on the indicated intervals:
(i) f (x) = cos 2 (x – π/4) on [0, π/2]

Solution:

Given function is $f(x) = \cos 2\left(x - \frac{\pi}{4}\right)$ on $\left[0, \frac{\pi}{2}\right]$

We know that cosine function is continuous and differentiable on R.

Let's find the values of the function at an extreme,

⇒ $f(0) = \cos 2\left(0 - \frac{\pi}{4}\right)$

⇒ $f(0) = \cos 2\left(-\frac{\pi}{4}\right)$

⇒ $f(0) = \cos\left(-\frac{\pi}{2}\right)$

We know that cos (– x) = cos x

⇒ f (0) = 0

⇒ $f\left(\frac{\pi}{2}\right) = \cos 2\left(\frac{\pi}{2} - \frac{\pi}{4}\right)$

⇒ $f\left(\frac{\pi}{2}\right) = \cos 2\left(\frac{\pi}{4}\right)$

$\Rightarrow f\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right)$

$\Rightarrow f\left(\frac{\pi}{2}\right) = 0$

We get $f(0) = f\left(\frac{\pi}{2}\right)$, so there exist a $c \in \left(0, \frac{\pi}{2}\right)$ such that f'(c) = 0.

Let's find the derivative of f(x)

$\Rightarrow f'(x) = \frac{d\left(\cos 2\left(x-\frac{\pi}{4}\right)\right)}{dx}$

$\Rightarrow f'(x) = -\sin\left(2\left(x-\frac{\pi}{4}\right)\right)\frac{d\left(2\left(x-\frac{\pi}{4}\right)\right)}{dx}$

$\Rightarrow f'(x) = -2\sin 2\left(x-\frac{\pi}{4}\right)$

We have f'(c) = 0,

$\Rightarrow -2\sin 2\left(c-\frac{\pi}{4}\right) = 0$

$\Rightarrow c-\frac{\pi}{4} = 0$

$\Rightarrow c = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$

∴ Rolle's Theorem is verified.

(ii) f (x) = sin 2x on [0, π/2]

Solution:

Given function is f (x) = sin2x on $\left[0, \frac{\pi}{2}\right]$

We know that sine function is continuous and differentiable on R. Let's find the values of function at extreme,

⇒ f (0) = sin2 (0)

⇒ f (0) = sin0

⇒ f (0) = 0

$\Rightarrow f\left(\frac{\pi}{2}\right) = \sin 2\left(\frac{\pi}{2}\right)$

$\Rightarrow f\left(\frac{\pi}{2}\right) = \sin(\pi)$

$\Rightarrow f\left(\frac{\pi}{2}\right) = 0$

We have $f(0) = f\left(\frac{\pi}{2}\right)$, so there exist a $c \in \left(0, \frac{\pi}{2}\right)$ such that f'(c) = 0.

Let's find the derivative of f(x)

$\Rightarrow f'(x) = \frac{d(\sin 2x)}{dx}$

$\Rightarrow f'(x) = \cos 2x \frac{d(2x)}{dx}$

⇒ f'(x) = 2cos2x

We have f'(c) = 0,

⇒ 2 cos 2c = 0

$\Rightarrow 2c = \frac{\pi}{2}$

$\Rightarrow c = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$

∴ Rolle's Theorem is verified.

(iii) f (x) = cos 2x on [-π/4, π/4]

Solution:

Given function is cos 2x on $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

We know that cosine function is continuous and differentiable on R. Let's find the values of the function at an extreme,

$\Rightarrow f\left(-\frac{\pi}{4}\right) = \cos 2\left(-\frac{\pi}{4}\right)$

$\Rightarrow f(0) = \cos\left(-\frac{\pi}{2}\right)$

We know that cos (– x) = cos x

⇒ f (0) = 0

$\Rightarrow f\left(\frac{\pi}{4}\right) = \cos 2\left(\frac{\pi}{4}\right)$

$\Rightarrow f\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right)$

$\Rightarrow f\left(\frac{\pi}{2}\right) = 0$

We have $f\left(-\frac{\pi}{4}\right) = f\left(\frac{\pi}{4}\right)$, so there exist a $c \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ such that f′(c) = 0.

Let's find the derivative of f(x)

$\Rightarrow f'(x) = \frac{d(\cos 2x)}{dx}$

$\Rightarrow f'(x) = -\sin 2x \frac{d(2x)}{dx}$

⇒ f′(x) = – 2sin2x

We have f′(c) = 0,

⇒ – 2sin2c = 0

⇒ 2c = 0

$\Rightarrow c = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$

∴ Rolle's Theorem is verified.

(iv) f (x) = e^x sin x on [0, π]

Solution:

Given function is f (x) = e^x sin x on [0, π]

We know that exponential and sine functions are continuous and differentiable on R.
Let's find the values of the function at an extreme,

$\Rightarrow f(0) = e^0 \sin(0)$

$\Rightarrow f(0) = 1 \times 0$

$\Rightarrow f(0) = 0$

$\Rightarrow f(\pi) = e^{\pi} \sin(\pi)$

$\Rightarrow f(\pi) = e^{\pi} \times 0$

$\Rightarrow f(\pi) = 0$

We have $f(0) = f(\pi)$, so there exist a $c \in (0, \pi)$ such that $f'(c) = 0$.

Let's find the derivative of f(x)

$\Rightarrow f'(x) = \frac{d(e^x \sin x)}{dx}$

$\Rightarrow f'(x) = \sin x \frac{d(e^x)}{dx} + e^x \frac{d(\sin x)}{dx}$

$\Rightarrow f'(x) = e^x (\sin x + \cos x)$

We have $f'(c) = 0$,

$\Rightarrow e^c (\sin c + \cos c) = 0$

$\Rightarrow \sin c + \cos c = 0$

$\Rightarrow \frac{1}{\sqrt{2}} \sin c + \frac{1}{\sqrt{2}} \cos c = 0$

$\Rightarrow \sin\left(\frac{\pi}{4}\right) \sin c + \cos\left(\frac{\pi}{4}\right) \cos c = 0$

$\Rightarrow \cos\left(c - \frac{\pi}{4}\right) = 0$

$\Rightarrow c - \frac{\pi}{4} = \frac{\pi}{2}$

$\Rightarrow c = \frac{3\pi}{4} \in (0, \pi)$

∴ Rolle's Theorem is verified.

(v) f (x) = e^x cos x on [-π/2, π/2]

Solution:

Given function is f(x) = e^x cos x on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

We know that exponential and cosine functions are continuous and differentiable on R. Let's find the values of the function at an extreme,

$\Rightarrow f\left(-\frac{\pi}{2}\right) = e^{-\frac{\pi}{2}} \cos\left(-\frac{\pi}{2}\right)$

$\Rightarrow f\left(-\frac{\pi}{2}\right) = e^{-\frac{\pi}{2}} \times 0$

$\Rightarrow f\left(-\frac{\pi}{2}\right) = 0$

$\Rightarrow f\left(\frac{\pi}{2}\right) = e^{\frac{\pi}{2}} \cos\left(\frac{\pi}{2}\right)$

$\Rightarrow f(\pi) = e^{\frac{\pi}{2}} \times 0$

$\Rightarrow f(\pi) = 0$

We have $f\left(-\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}\right)$, so there exist a $c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that f'(c) = 0.

Let's find the derivative of f(x)

$\Rightarrow f'(x) = \frac{d(e^x \cos x)}{dx}$

$\Rightarrow f'(x) = \cos x \frac{d(e^x)}{dx} + e^x \frac{d(\cos x)}{dx}$

$\Rightarrow f'(x) = e^x(-\sin x + \cos x)$

We have f'(c) = 0,

$\Rightarrow e^c(-\sin c + \cos c) = 0$

$\Rightarrow -\sin c + \cos c = 0$

$\Rightarrow \frac{-1}{\sqrt{2}} \sin c + \frac{1}{\sqrt{2}} \cos c = 0$

$\Rightarrow -\sin\left(\frac{\pi}{4}\right)\text{sinc} + \cos\left(\frac{\pi}{4}\right)\text{cosc} = 0$

$\Rightarrow \cos\left(c + \frac{\pi}{4}\right) = 0$

$\Rightarrow c + \frac{\pi}{4} = \frac{\pi}{2}$

$\Rightarrow c = \frac{\pi}{4} \epsilon \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

∴ Rolle's Theorem is verified.

(vi) f (x) = cos 2x on [0, π]

Solution:
Given function is f (x) = cos 2x on [0, π]
We know that cosine function is continuous and differentiable on R. Let's find the values of function at extreme,
⇒ f (0) = cos2(0)
⇒ f (0) = cos(0)
⇒ f (0) = 1
⇒ f (π) = cos2(π)
⇒ f (π) = cos(2 π)
⇒ f (π) = 1
We have f (0) = f (π), so there exist a c belongs to (0, π) such that f'(c) = 0.
Let's find the derivative of f(x)

$\Rightarrow f'(x) = \frac{d(\cos 2x)}{dx}$

$\Rightarrow f'(x) = -\sin 2x \frac{d(2x)}{dx}$

⇒ f'(x) = – 2sin2x

We have f'(c) = 0,

⇒ – 2sin2c = 0

⇒ 2c = 0

$\Rightarrow c = \frac{\pi}{4} \epsilon (0, \pi)$

Hence Rolle's Theorem is verified.

(vii) $f(x) = \frac{\sin x}{e^x}$ **on** $0 \le x \le \pi$

Solution:

Given function is $f(x) = \frac{\sin x}{e^x}$ on $[0, \pi]$

This can be written as

$\Rightarrow f(x) = e^{-x} \sin x$ on $[0, \pi]$

We know that exponential and sine functions are continuous and differentiable on R. Let's find the values of the function at an extreme,

$\Rightarrow f(0) = e^{-0}\sin(0)$

$\Rightarrow f(0) = 1\times 0$

$\Rightarrow f(0) = 0$

$\Rightarrow f(\pi) = e^{-\pi}\sin(\pi)$

$\Rightarrow f(\pi) = e^{-\pi} \times 0$

$\Rightarrow f(\pi) = 0$

We have f (0) = f (π), so there exist a c belongs to (0, π) such that f'(c) = 0.

Let's find the derivative of f(x)

$\Rightarrow f'(x) = \frac{d(e^{-x}\sin x)}{dx}$

$\Rightarrow f'(x) = \sin x \frac{d(e^{-x})}{dx} + e^{-x}\frac{d(\sin x)}{dx}$

$\Rightarrow f'(x) = \sin x(-e^{-x}) + e^{-x}(\cos x)$

$\Rightarrow f'(x) = e^{-x}(-\sin x + \cos x)$

We have f'(c) = 0,

$\Rightarrow e^{-c}(-\sin c + \cos c) = 0$

$\Rightarrow -\sin c + \cos c = 0$

$\Rightarrow -\frac{1}{\sqrt{2}}\sin c + \frac{1}{\sqrt{2}}\cos c = 0$

$\Rightarrow -\sin\left(\frac{\pi}{4}\right)\sin c + \cos\left(\frac{\pi}{4}\right)\cos c = 0$

$\Rightarrow \cos\left(c + \frac{\pi}{4}\right) = 0$

$\Rightarrow c + \frac{\pi}{4} = \frac{\pi}{2}$

$\Rightarrow c = \frac{\pi}{4} \epsilon (0, \pi)$

∴ Rolle's Theorem is verified.

(viii) f (x) = sin 3x on [0, π]

Solution:
Given function is f (x) = sin3x on [0, π]
We know that sine function is continuous and differentiable on R. Let's find the values of function at extreme,
⇒ f (0) = sin3(0)
⇒ f (0) = sin0
⇒ f (0) = 0
⇒ f (π) = sin3(π)
⇒ f (π) = sin(3 π)
⇒ f (π) = 0
We have f (0) = f (π), so there exist a c belongs to (0, π) such that f'(c) = 0.
Let's find the derivative of f(x)

$\Rightarrow f'(x) = \frac{d(\sin 3x)}{dx}$

$\Rightarrow f'(x) = \cos 3x \frac{d(3x)}{dx}$

⇒ f'(x) = 3cos3x

We have f'(c) = 0,

$\Rightarrow$ 3cos3c = 0

$\Rightarrow 3c = \frac{\pi}{2}$

$\Rightarrow c = \frac{\pi}{6} \epsilon (0, \pi)$

∴ Rolle's Theorem is verified.

(ix) $f(x) = e^{1-x^2}$ on $[-1, 1]$

Solution:

Given function is $f(x) = e^{1-x^2}$ on [– 1, 1]

We know that exponential function is continuous and differentiable over R.
Let's find the value of function f at extremes,

$\Rightarrow f(-1) = e^{1-(-1)^2}$

$\Rightarrow f(-1) = e^{1-1}$

$\Rightarrow$ f (– 1) = e^0

$\Rightarrow$ f (– 1) = 1

$\Rightarrow f(1) = e^{1-1^2}$

$\Rightarrow f(1) = e^{1-1}$

$\Rightarrow$ f (1) = e^0

$\Rightarrow$ f (1) = 1

We got f (– 1) = f (1) so, there exists a c ϵ (– 1, 1) such that f'(c) = 0.

Let's find the derivative of the function f:

$\Rightarrow f'(x) = \frac{d\left(e^{1-x^2}\right)}{dx}$

$\Rightarrow f'(x) = e^{1-x^2}\frac{d(1-x^2)}{dx}$

$\Rightarrow f'(x) = e^{1-x^2}(-2x)$

We have f′(c) = 0

$\Rightarrow e^{1-c^2}(-2c) = 0$

⇒ 2c = 0

⇒ c = 0 ϵ [– 1, 1]

∴ Rolle’s Theorem is verified.

(x) f (x) = log (x^2 + 2) – log 3 on [-1, 1]

Solution:

Given function is f (x) = log(x^2 + 2) – log3 on [– 1, 1]
We know that logarithmic function is continuous and differentiable in its own domain.
We check the values of the function at the extreme,
⇒ f (– 1) = log(($-1)^2$ + 2) – log 3
⇒ f (– 1) = log (1 + 2) – log 3
⇒ f (– 1) = log 3 – log 3
⇒ f (– 1) = 0
⇒ f (1) = log (1^2 + 2) – log 3
⇒ f (1) = log (1 + 2) – log 3
⇒ f (1) = log 3 – log 3
⇒ f (1) = 0
We have got f (– 1) = f (1). So, there exists a c such that c ϵ (– 1, 1) such that f′(c) = 0.
Let’s find the derivative of the function f,

$\Rightarrow f'(x) = \frac{d(\log(x^2+2)-\log 3)}{dx}$

$\Rightarrow f'(x) = \frac{1}{x^2+2}\frac{d(x^2+2)}{dx} - 0$

$\Rightarrow f'(x) = \frac{2x}{x^2+2}$

We have f'(c) = 0

$\Rightarrow \frac{2c}{c^2+2} = 0$

⇒ 2c = 0

⇒ c = 0 ϵ (– 1, 1)

∴ Rolle's Theorem is verified.

(xi) f (x) = sin x + cos x on [0, π/2]

Solution:

Given function is f(x) = sin x + cos x on $\left[0, \frac{\pi}{2}\right]$

We know that sine and cosine functions are continuous and differentiable on R. Let's the value of function f at extremes:

⇒ f (0) = sin (0) + cos (0)

⇒ f (0) = 0 + 1

⇒ f (0) = 1

$\Rightarrow f\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right)$

$\Rightarrow f\left(\frac{\pi}{2}\right) = 1 + 0$

$\Rightarrow f\left(\frac{\pi}{2}\right) = 1$

We have $f(0) = f\left(\frac{\pi}{2}\right)$. So, there exists a c ϵ $\left(0, \frac{\pi}{2}\right)$ such that f'(c) = 0.

Let's find the derivative of the function f.

$\Rightarrow f'(x) = \frac{d(\sin x + \cos x)}{dx}$

⇒ f'(x) = cos x – sin x

$\Rightarrow f'(x) = 4\cos^2 x + 2\cos x - 2$
We have $f'(c) = 0$,
$\Rightarrow 4\cos^2 c + 2\cos c - 2 = 0$
$\Rightarrow 2\cos^2 c + \cos c - 1 = 0$
$\Rightarrow 2\cos^2 c + 2\cos c - \cos c - 1 = 0$
$\Rightarrow 2\cos c(\cos c + 1) - 1(\cos c + 1) = 0$
$\Rightarrow (2\cos c - 1)(\cos c + 1) = 0$

$\Rightarrow \cos c = \frac{1}{2}$ or $\cos c = -1$

$\Rightarrow c = \frac{\pi}{3} \in (0, \pi)$

∴ Rolle's Theorem is verified.

(xiii) $f(x) = \frac{x}{2} - \sin\frac{\pi x}{6}$ **on** $[-1, 0]$

Solution:

Given function is $f(x) = \frac{x}{2} - \sin\left(\frac{\pi x}{6}\right)$ on $[-1, 0]$

We know that sine function is continuous and differentiable over R.

Now we have to check the values of 'f' at an extreme

$\Rightarrow f(-1) = \frac{-1}{2} - \sin\left(\frac{\pi(-1)}{6}\right)$

$\Rightarrow f(-1) = -\frac{1}{2} - \sin\left(\frac{-\pi}{6}\right)$

$\Rightarrow f(-1) = -\frac{1}{2} - \left(-\frac{1}{2}\right)$

$\Rightarrow f(-1) = 0$

$\Rightarrow f(0) = \frac{0}{2} - \sin\left(\frac{\pi(0)}{6}\right)$

$\Rightarrow f(0) = 0 - \sin(0)$

$\Rightarrow f(0) = 0 - 0$

⇒ f (0) = 0

We have got f (– 1) = f (0). So, there exists a c ϵ (– 1, 0) such that f'(c) = 0.

Now we have to find the derivative of the function 'f'

$\Rightarrow f'(x) = \frac{d\left(\frac{x}{2} - \sin\left(\frac{\pi x}{6}\right)\right)}{dx}$

$\Rightarrow f'(x) = \frac{1}{2} - \cos\left(\frac{\pi x}{6}\right)\frac{d\left(\frac{\pi x}{6}\right)}{dx}$

$\Rightarrow f'(x) = \frac{1}{2} - \frac{\pi}{6}\cos\left(\frac{\pi x}{6}\right)$

We have f'(c) = 0

$\Rightarrow \frac{1}{2} - \frac{\pi}{6}\cos\left(\frac{\pi c}{6}\right) = 0$

$\Rightarrow \frac{\pi}{6}\cos\left(\frac{\pi c}{6}\right) = \frac{1}{2}$

$\Rightarrow \cos\left(\frac{\pi c}{6}\right) = \frac{1}{2} \times \frac{6}{\pi}$

$\Rightarrow \cos\left(\frac{\pi c}{6}\right) = \frac{3}{\pi}$

$\Rightarrow \frac{\pi c}{6} = \cos^{-1}\left(\frac{3}{\pi}\right)$

$\Rightarrow c = \frac{6}{\pi}\cos^{-1}\left(\frac{3}{\pi}\right)$

Cosine is positive between $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, for our convenience we take the interval to be $-\frac{\pi}{2} \le \theta \le 0$, since the values of the cosine repeats.

We know that $\frac{3}{\pi}$ value is nearly equal to 1. So, the value of the c nearly equal to 0.

So, we can clearly say that cϵ (– 1, 0).

∴ Rolle's Theorem is verified.

$\Rightarrow f'(x) = \frac{6}{\pi} - 4(2\sin x\cos x)$

$\Rightarrow f'(x) = \frac{6}{\pi} - 4\sin 2x$

We have f'(c) = 0

$\Rightarrow \frac{6}{\pi} - 4\sin 2c = 0$

$\Rightarrow 4\sin 2c = \frac{6}{\pi}$

$\Rightarrow \sin 2c = \frac{6}{4\pi}$

We know $\frac{6}{4\pi} < \frac{1}{2}$

$\Rightarrow \sin 2c < \frac{1}{2}$

$\Rightarrow 2c < \sin^{-1}\left(\frac{1}{2}\right)$

$\Rightarrow 2c < \frac{\pi}{6}$

$\Rightarrow c < \frac{\pi}{12} \epsilon \left(0, \frac{\pi}{6}\right)$

∴ Rolle's Theorem is verified.

(xv) f (x) = $4^{\sin x}$ on [0, π]

Solution:

Given function is f (x) = $4^{\sin x}$ on [0, π]

We that sine function is continuous and differentiable over R.

Now we have to check the values of function 'f' at extremes

⇒ f (0) = $4^{\sin (0)}$

⇒ f (0) = 4^0

⇒ f (0) = 1

(xiv). $f(x) = \frac{6x}{\pi} - 4\sin^2 x$ **on** $[0, \frac{\pi}{6}]$

Solution:

Given function is $f(x) = \frac{6x}{\pi} - 4\sin^2 x$ on $\left[0, \frac{\pi}{6}\right]$

We know that sine function is continuous and differentiable over R.

Now we have to check the values of function 'f' at the extremes,

$\Rightarrow f(0) = \frac{6(0)}{\pi} - 4\sin^2(0)$

$\Rightarrow$ f (0) = 0 – 4(0)

$\Rightarrow$ f (0) = 0

$\Rightarrow f\left(\frac{\pi}{6}\right) = \frac{6\left(\frac{\pi}{6}\right)}{\pi} - 4\sin^2\left(\frac{\pi}{6}\right)$

$\Rightarrow f\left(\frac{\pi}{6}\right) = \frac{\pi}{\pi} - 4\left(\frac{1}{2}\right)^2$

$\Rightarrow f\left(\frac{\pi}{6}\right) = 1 - 4\left(\frac{1}{4}\right)$

$\Rightarrow f\left(\frac{\pi}{6}\right) = 1 - 1$

$\Rightarrow f\left(\frac{\pi}{6}\right) = 0$.

We have $f(0) = f\left(\frac{\pi}{6}\right)$. So, there exists a c $\in \left(0, \frac{\pi}{6}\right)$ such that f'(c) = 0.

We have to find the derivative of function 'f.'

$\Rightarrow f'(x) = \frac{d\left(\frac{6x}{\pi} - 4\sin^2 x\right)}{dx}$

$\Rightarrow f'(x) = \frac{6}{\pi} - 4 \times 2\sin x \times \frac{d(\sin x)}{dx}$

$\Rightarrow f'(x) = \frac{6}{\pi} - 8\sin x(\cos x)$

$\Rightarrow f'(x) = \frac{6}{\pi} - 4(2\sin x\cos x)$

$\Rightarrow f'(x) = \frac{6}{\pi} - 4\sin 2x$

We have f'(c) = 0

$\Rightarrow \frac{6}{\pi} - 4\sin 2c = 0$

$\Rightarrow 4\sin 2c = \frac{6}{\pi}$

$\Rightarrow \sin 2c = \frac{6}{4\pi}$

We know $\frac{6}{4\pi} < \frac{1}{2}$

$\Rightarrow \sin 2c < \frac{1}{2}$

$\Rightarrow 2c < \sin^{-1}\left(\frac{1}{2}\right)$

$\Rightarrow 2c < \frac{\pi}{6}$

$\Rightarrow c < \frac{\pi}{12} \epsilon \left(0, \frac{\pi}{6}\right)$

∴ Rolle's Theorem is verified.

(xv) f (x) = $4^{\sin x}$ on [0, π]

Solution:
Given function is f (x) = $4^{\sin x}$ on [0, π]

We that sine function is continuous and differentiable over R.

Now we have to check the values of function 'f' at extremes

⇒ f (0) = $4^{\sin (0)}$

⇒ f (0) = 4^0

⇒ f (0) = 1

⇒ $f(\pi) = 4^{\sin\pi}$

⇒ $f(\pi) = 4^0$

⇒ $f(\pi) = 1$

We have f (0) = f (π). So, there exists a c ϵ (0, π) such that f'(c) = 0.

Now we have to find the derivative of 'f'

⇒ $f'(x) = \frac{d(4^{\sin x})}{dx}$

⇒ $f'(x) = 4^{\sin x}\log 4\frac{d(\sin x)}{dx}$

⇒ $f'(x) = 4^{\sin x}\log 4\cos x$

We have f'(c) = 0

⇒ $4^{\sin c}\log 4\cos c = 0$

⇒ Cos c = 0

⇒ $c = \frac{\pi}{2}\epsilon(0,\pi)$

∴ Rolle's Theorem is verified.

(xvi) f (x) = x^2 – 5x + 4 on [0, π/6]

Solution:
Given function is f (x) = x^2 – 5x + 4 on [1, 4]
Since, given function f is a polynomial it is continuous and differentiable everywhere i.e., on R.
Let us find the values at extremes
⇒ f (1) = 1^2 – 5(1) + 4
⇒ f (1) = 1 – 5 + 4
⇒ f (1) = 0
⇒ f (4) = 4^2 – 5(4) + 4
⇒ f (4) = 16 – 20 + 4
⇒ f (4) = 0
We have f (1) = f (4). So, there exists a c ϵ (1, 4) such that f'(c) = 0.

Let's find the derivative of f(x):

$\Rightarrow f'(x) = \frac{d(x^2 - 5x + 4)}{dx}$

$\Rightarrow f'(x) = \frac{d(x^2)}{dx} - \frac{d(5x)}{dx} + \frac{d(4)}{dx}$

$\Rightarrow f'(x) = 2x - 5 + 0$

$\Rightarrow f'(x) = 2x - 5$

We have f'(c) = 0

$\Rightarrow f'(c) = 0$

$\Rightarrow 2c - 5 = 0$

$\Rightarrow 2c = 5$

$\Rightarrow c = \frac{5}{2}$

$\Rightarrow C = 2.5 \in (1, 4)$

∴ Rolle's Theorem is verified.

(xvii) f (x) = $\sin^4 x + \cos^4 x$ on [0, π/2]

Solution:

Given function is f (x) = $\sin^4 x + \cos^4 x$ on $\left[0, \frac{\pi}{2}\right]$

We know that sine and cosine functions are continuous and differentiable functions over R.

Now we have to find the value of function 'f' at extremes

$\Rightarrow f(0) = \sin^4(0) + \cos^4(0)$

$\Rightarrow f(0) = (0)^4 + (1)^4$

$\Rightarrow f(0) = 0 + 1$

$\Rightarrow f(0) = 1$

$\Rightarrow f\left(\frac{\pi}{2}\right) = \sin^4\left(\frac{\pi}{2}\right) + \cos^4\left(\frac{\pi}{2}\right)$

$\Rightarrow f\left(\frac{\pi}{2}\right) = 1^4 + 0^4$

$\Rightarrow f\left(\frac{\pi}{2}\right) = 1 + 0$

$\Rightarrow f\left(\frac{\pi}{2}\right) = 1$

We have $f(0) = f\left(\frac{\pi}{2}\right)$. So, there exists a c $\epsilon\left(0, \frac{\pi}{2}\right)$ such that f′(c) = 0.

Now we have to find the derivative of the function ‘f’.

$\Rightarrow f'(x) = \frac{d(\sin^4 x + \cos^4 x)}{dx}$

$\Rightarrow f'(x) = 4\sin^3 x \frac{d(\sin x)}{dx} + 4\cos^3 x \frac{d(\cos x)}{dx}$

$\Rightarrow f'(x) = 4\sin^3 x\cos x - 4\cos^3 x\sin x$

$\Rightarrow f'(x) = 4 \sin x \cos x (\sin^2 x - \cos^2 x)$

$\Rightarrow f'(x) = 2(2 \sin x \cos x) (- \cos 2x)$

$\Rightarrow f'(x) = - 2(\sin 2x) (\cos 2x)$

$\Rightarrow f'(x) = - \sin 4x$

We have f′(c) = 0

$\Rightarrow - \sin 4c = 0$

$\Rightarrow \sin 4c = 0$

$\Rightarrow 4c = 0$ or π

$\Rightarrow c = \frac{\pi}{4} \epsilon \left(0, \frac{\pi}{2}\right)$

∴ Rolle’s Theorem is verified.

(xviii) f (x) = sin x – sin 2x on [0, π]

Solution:

Given function is f (x) = sin x – sin2x on [0, π]

We know that sine function is continuous and differentiable over R.

Now we have to check the values of the function 'f' at the extremes.

⇒ f (0) = sin (0)–sin 2(0)

⇒ f (0) = 0 – sin (0)

⇒ f (0) = 0

⇒ f (π) = sin(π) – sin2(π)

⇒ f (π) = 0 – sin(2π)

⇒ f (π) = 0

We have f (0) = f (π). So, there exists a c ϵ (0, π) such that f'(c) = 0.

Now we have to find the derivative of the function 'f'

$\Rightarrow f'(x) = \frac{d(\sin x - \sin 2x)}{dx}$

$\Rightarrow f'(x) = \cos x - \cos 2x \frac{d(2x)}{dx}$

$\Rightarrow f'(x) = \cos x - 2\cos 2x$

⇒ $f'(x) = \cos x - 2(2\cos^2 x - 1)$

⇒ $f'(x) = \cos x - 4\cos^2 x + 2$

We have f'(c) = 0

⇒ $\cos c - 4\cos^2 c + 2 = 0$

$\Rightarrow \cos c = \frac{-1 \pm \sqrt{(1)^2 - (4 \times -4 \times 2)}}{2 \times -4}$

$\Rightarrow \cos c = \frac{-1 \pm \sqrt{1 + 33}}{-8}$

$\Rightarrow c = \cos^{-1}\left(\frac{-1 \pm \sqrt{33}}{-8}\right)$

We can see that c ϵ (0, π)

∴ Rolle's Theorem is verified.

4. Using Rolle's Theorem, find points on the curve $y = 16 - x^2$, x ∈ [-1, 1], where tangent is parallel to x – axis.

Solution:

Given function is $y = 16 - x^2$, $x \in [-1, 1]$

We know that polynomial function is continuous and differentiable over R.

Let us check the values of 'y' at extremes

$\Rightarrow y(-1) = 16 - (-1)^2$

$\Rightarrow y(-1) = 16 - 1$

$\Rightarrow y(-1) = 15$

$\Rightarrow y(1) = 16 - (1)^2$

$\Rightarrow y(1) = 16 - 1$

$\Rightarrow y(1) = 15$

We have y (– 1) = y (1). So, there exists a $c \in (-1, 1)$ such that $f'(c) = 0$.

We know that for a curve g, the value of the slope of the tangent at a point r is given by g'(r).

Now we have to find the derivative of curve y

$\Rightarrow y' = \frac{d(16-x^2)}{dx}$

$\Rightarrow y' = -2x$

We have $y'(c) = 0$

$\Rightarrow -2c = 0$

$\Rightarrow c = 0 \in (-1, 1)$

Value of y at x = 1 is

$\Rightarrow y = 16 - 0^2$

$\Rightarrow y = 16$

∴ The point at which the curve y has a tangent parallel to x – axis (since the slope of x – axis is 0) is (0, 16).

EXERCISE 15.2 PAGE NO: 15.17

1. Verify Lagrange's mean value theorem for the following functions on the indicated intervals. In each case find a point 'c' in the indicated interval as stated by the Lagrange's mean value theorem:
(i) f (x) = $x^2 - 1$ on [2, 3]

Solution:

Given f (x) = $x^2 - 1$ on [2, 3]

We know that every polynomial function is continuous everywhere on (−∞, ∞) and differentiable for all arguments. Here, f(x) is a polynomial function. So it is continuous in [2, 3] and differentiable in (2, 3). So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exist a point c ∈ (2, 3) such that:

$$f'(c) = \frac{f(3) - f(2)}{3 - 2}$$

$$\Rightarrow f'(c) = \frac{f(3) - f(2)}{1}$$

f (x) = $x^2 - 1$

Differentiating with respect to x

f'(x) = 2x

For f'(c), put the value of x=c in f'(x):

f'(c) = 2c

For f (3), put the value of x=3 in f(x):

f (3) = $(3)^2 - 1$

= 9 − 1

= 8

For f (2), put the value of x=2 in f(x):

$f(2) = (2)^2 - 1$

$= 4 - 1$

$= 3$

$\therefore f'(c) = f(3) - f(2)$

$\Rightarrow 2c = 8 - 3$

$\Rightarrow 2c = 5$

$\Rightarrow c = \frac{5}{2} \in (2, 3)$

Hence, Lagrange's mean value theorem is verified.

(ii) f (x) = $x^3 - 2x^2 - x + 3$ on [0, 1]

Solution:

Given f (x) = $x^3 - 2x^2 - x + 3$ on [0, 1]
Every polynomial function is continuous everywhere on (−∞, ∞) and differentiable for all arguments. Here, f(x) is a polynomial function. So it is continuous in [0, 1] and differentiable in (0, 1). So both the necessary conditions of Lagrange's mean value theorem is satisfied.
Therefore, there exist a point c ∈ (0, 1) such that:

$$f'(c) = \frac{f(1) - f(0)}{1 - 0}$$

$$\Rightarrow f'(c) = \frac{f(1) - f(0)}{1}$$

f (x) = $x^3 - 2x^2 - x + 3$
Differentiating with respect to x
$f'(x) = 3x^2 - 2(2x) - 1$
$= 3x^2 - 4x - 1$
For f'(c), put the value of x=c in f'(x)
$f'(c) = 3c^2 - 4c - 1$
For f (1), put the value of x = 1 in f(x)
$f(1) = (1)^3 - 2(1)^2 - (1) + 3$

= 1 – 2 – 1 + 3
= 1
For f (0), put the value of x=0 in f(x)
f (0)= $(0)^3 - 2(0)^2 - (0) + 3$
= 0 – 0 – 0 + 3
= 3
∴ f′(c) = f(1) – f(0)
⇒ $3c^2 - 4c - 1 = 1 - 3$
⇒ $3c^2 - 4c = 1 + 1 - 3$
⇒ $3c^2 - 4c = -1$
⇒ $3c^2 - 4c + 1 = 0$
⇒ $3c^2 - 3c - c + 1 = 0$
⇒ 3c(c – 1) – 1(c – 1) = 0
⇒ (3c – 1) (c – 1) = 0

$$\Rightarrow c = \frac{1}{3}, 1$$

$$\Rightarrow c = \frac{1}{3} \in (0, 1)$$

Hence, Lagrange's mean value theorem is verified.

(iii) f (x) = x (x – 1) on [1, 2]

Solution:
Given f (x) = x (x – 1) on [1, 2]
= $x^2 - x$
Every polynomial function is continuous everywhere on (−∞, ∞) and differentiable for all arguments. Here, f(x) is a polynomial function. So it is continuous in [1, 2] and differentiable in (1, 2). So both the necessary conditions of Lagrange's mean value theorem is satisfied.
Therefore, there exist a point c ∈ (1, 2) such that:

$$f'(c) = \frac{f(2) - f(1)}{2 - 1}$$

$$\Rightarrow f'(c) = \frac{f(2) - f(1)}{1}$$

f (x) = $x^2 - x$

Differentiating with respect to x

$f'(x) = 2x - 1$

For f'(c), put the value of x=c in f'(x):

$f'(c)= 2c - 1$

For f (2), put the value of x = 2 in f(x)

$f(2) = (2)^2 - 2$

$= 4 - 2$

$= 2$

For f (1), put the value of x = 1 in f(x):

$f(1)= (1)^2 - 1$

$= 1 - 1$

$= 0$

$\therefore f'(c) = f(2) - f(1)$

$\Rightarrow 2c - 1 = 2 - 0$

$\Rightarrow 2c = 2 + 1$

$\Rightarrow 2c = 3$

$\Rightarrow c = \frac{3}{2} \in (1, 2)$

Hence, Lagrange's mean value theorem is verified.

(iv) f (x) = $x^2 - 3x + 2$ on [-1, 2]

Solution:

Given f (x) = $x^2 - 3x + 2$ on [– 1, 2]

Every polynomial function is continuous everywhere on (−∞, ∞) and differentiable for all arguments. Here, f(x) is a polynomial function. So it is continuous in [– 1, 2] and differentiable in (– 1, 2). So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exist a point c ∈ (-1, 2) such that:

$$f'(c) = \frac{f(2) - f(-1)}{2 - (-1)}$$

$$\Rightarrow f'(c) = \frac{f(2) - f(-1)}{2 + 1}$$

$$\Rightarrow f'(c) = \frac{f(2) - f(-1)}{3}$$

f (x) = $x^2 - 3x + 2$
Differentiating with respect to x
f'(x) = 2x – 3
For f'(c), put the value of x = c in f'(x):
f'(c)= 2c – 3
For f (2), put the value of x = 2 in f(x)
f (2) = $(2)^2 - 3(2) + 2$
= 4 – 6 + 2
= 0
For f (– 1), put the value of x = – 1 in f(x):
f (– 1) = $(-1)^2 - 3(-1) + 2$
= 1 + 3 + 2
= 6

$$f'(c) = \frac{f(2) - f(-1)}{3}$$

$$\Rightarrow 2c - 3 = \frac{0 - 6}{3}$$

$$\Rightarrow 2c = \frac{-6}{3} + 3$$

$$\Rightarrow 2c = -2 + 3$$

$$\Rightarrow 2c = -1$$

$$\Rightarrow c = \frac{-1}{2} \in (-1, 2)$$

Hence, Lagrange's mean value theorem is verified.

(v) f (x) = $2x^2 - 3x + 1$ on [1, 3]

Solution:
Given f (x) = $2x^2 - 3x + 1$ on [1, 3]
Every polynomial function is continuous everywhere on (−∞, ∞) and differentiable for all arguments. Here, f(x) is a polynomial function. So it is continuous in [1, 3] and differentiable in (1, 3). So both the necessary conditions of Lagrange's mean value theorem is satisfied.
Therefore, there exist a point c ∈ (1, 3) such that:

$$f'(c)=\frac{f(3)-f(1)}{3-1}$$

$$\Rightarrow f'(c)=\frac{f(3)-f(1)}{2}$$

$f(x) = 2x^2 - 3x + 1$

Differentiating with respect to x

$f'(x) = 2(2x) - 3$

$= 4x - 3$

For f'(c), put the value of x = c in f'(x):

$f'(c)= 4c - 3$

For f (3), put the value of x = 3 in f(x):

$f(3) = 2(3)^2 - 3(3) + 1$

$= 2(9) - 9 + 1$

$= 18 - 8 = 10$

For f (1), put the value of x = 1 in f(x):

$f(1) = 2(1)^2 - 3(1) + 1$

$= 2(1) - 3 + 1$

$= 2 - 2 = 0$

$$f'(c)=\frac{f(3)-f(1)}{2}$$

$$\Rightarrow 4c-3=\frac{10-0}{2}$$

$$\Rightarrow 4c=\frac{10}{2}+3$$

$$\Rightarrow 4c = 5 + 3$$

$$\Rightarrow 4c = 8$$

$$\Rightarrow c=\frac{8}{4}=2\in(1, 3)$$

Hence, Lagrange's mean value theorem is verified.

(vi) $f(x) = x^2 - 2x + 4$ on [1, 5]

Solution:

Given f (x) = $x^2 - 2x + 4$ on [1, 5]
Every polynomial function is continuous everywhere on (−∞, ∞) and differentiable for all arguments. Here, f(x) is a polynomial function. So it is continuous in [1, 5] and differentiable in (1, 5). So both the necessary conditions of Lagrange's mean value theorem is satisfied.
Therefore, there exist a point c ∈ (1, 5) such that:

$$f'(c)=\frac{f(5)-f(1)}{5-1}$$

$$\Rightarrow f'(c)=\frac{f(5)-f(1)}{4}$$

f (x) = $x^2 - 2x + 4$
Differentiating with respect to x:
f'(x) = 2x – 2
For f'(c), put the value of x=c in f'(x):
f'(c)= 2c – 2
For f (5), put the value of x=5 in f(x):
f (5)= $(5)^2 - 2(5) + 4$
= 25 – 10 + 4
= 19
For f (1), put the value of x = 1 in f(x)
f (1) = $(1)^2 - 2(1) + 4$
= 1 – 2 + 4
= 3

$$f'(c)=\frac{f(5)-f(1)}{4}$$

$$\Rightarrow 2c-2=\frac{19-3}{4}$$

$$\Rightarrow 2c=\frac{16}{4}+2$$

$$\Rightarrow 2c = 4 + 2$$

$$\Rightarrow 2c= 6$$

$$\Rightarrow c=\frac{6}{2}=3\in(1, 5)$$

Hence, Lagrange's mean value theorem is verified.

(vii) f (x) = 2x – x^2 on [0, 1]

Solution:

Given f (x) = 2x – x^2 on [0, 1]

Every polynomial function is continuous everywhere on (−∞, ∞) and differentiable for all arguments. Here, f(x) is a polynomial function. So it is continuous in [0, 1] and differentiable in (0, 1). So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exist a point c ∈ (0, 1) such that:

$$f'(c) = \frac{f(1) - f(0)}{1 - 0}$$

⇒ f'(c) = f(1) – f(0)

f (x) = 2x – x^2

Differentiating with respect to x:

f'(x) = 2 – 2x

For f'(c), put the value of x = c in f'(x):

f'(c)= 2 – 2c

For f (1), put the value of x = 1 in f(x):

f (1)= 2(1) – $(1)^2$

= 2 – 1

= 1

For f (0), put the value of x = 0 in f(x):

f (0) = 2(0) – $(0)^2$

= 0 – 0

= 0

f'(c) = f(1) – f(0)

⇒ 2 – 2c = 1 – 0

⇒ – 2c = 1 – 2

⇒ – 2c = – 1

$$\Rightarrow c = \frac{-1}{-2} = \frac{1}{2} \in (0, 1)$$

Hence, Lagrange's mean value theorem is verified.

(viii) f (x) = (x – 1) (x – 2) (x – 3)

Solution:

Given f (x) = (x – 1) (x – 2) (x – 3) on [0, 4]

$= (x^2 - x - 2x + 3)(x - 3)$
$= (x^2 - 3x + 3)(x - 3)$
$= x^3 - 3x^2 + 3x - 3x^2 + 9x - 9$
$= x^3 - 6x^2 + 12x - 9$ on [0, 4]
Every polynomial function is continuous everywhere on (−∞, ∞) and differentiable for all arguments. Here, f(x) is a polynomial function. So it is continuous in [0, 4] and differentiable in (0, 4). So both the necessary conditions of Lagrange's mean value theorem is satisfied.
Therefore, there exist a point c ∈ (0, 4) such that:

$$f'(c) = \frac{f(4) - f(0)}{4 - 0}$$

$$\Rightarrow f'(c) = \frac{f(4) - f(0)}{4}$$

$f(x) = x^3 - 6x^2 + 12x - 9$
Differentiating with respect to x:
$f'(x) = 3x^2 - 6(2x) + 12$
$= 3x^2 - 12x + 12$
For f'(c), put the value of x = c in f'(x):
$f'(c) = 3c^2 - 12c + 12$
For f (4), put the value of x = 4 in f(x):
$f(4) = (4)^3 - 6(4)^2 + 12(4) - 9$
$= 64 - 96 + 48 - 9$
$= 7$
For f (0), put the value of x = 0 in f(x):
$f(0) = (0)^3 - 6(0)^2 + 12(0) - 9$
$= 0 - 0 + 0 - 9$
$= -9$

$$f'(c) = \frac{f(4) - f(0)}{4}$$

$$\Rightarrow 3c^2 - 12c + 12 = \frac{7 - (-9)}{4}$$

$$\Rightarrow 3c^2 - 12c + 12 = \frac{7 + 9}{4}$$

$$\Rightarrow 3c^2 - 12c + 12 = \frac{16}{4}$$

$\Rightarrow 3c^2 - 12c + 12 = 4$

$\Rightarrow 3c^2 - 12c + 8 = 0$

We know that for quadratic equation, $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow c = \frac{-(-12) \pm \sqrt{(-12)^2 - 4 \times 3 \times 8}}{2 \times 3}$$

$$\Rightarrow c = \frac{12 \pm \sqrt{144 - 96}}{6}$$

$$\Rightarrow c = \frac{12 \pm \sqrt{48}}{6}$$

$$\Rightarrow c = \frac{12 \pm 4\sqrt{3}}{6}$$

$$\Rightarrow c = \frac{12}{6} \pm \frac{4\sqrt{3}}{6}$$

$$\Rightarrow c = 2 \pm \frac{2\sqrt{3}}{3}$$

$$\Rightarrow c = 2 + \frac{2\sqrt{3}}{3}, 2 - \frac{2\sqrt{3}}{3} \in c$$

Hence, Lagrange's mean value theorem is verified.

(ix). $f(x) = \sqrt{25 - x^2}$ on $[-3, 4]$

Solution:

Given

$f(x) = \sqrt{25 - x^2}$ on $[-3, 4]$

Here, $\sqrt{25 - x^2} > 0$

$\Rightarrow 25 - x^2 > 0$

$\Rightarrow x^2 < 25$

$\Rightarrow -5 < x < 5$

$\Rightarrow \sqrt{25 - x^2}$ has unique values for all $x \in (-5, 5)$

$\therefore$ f (x) is continuous in $[-3, 4]$

$$f(x) = (25 - x^2)^{\frac{1}{2}}$$

Differentiating with respect to x:

$$f'(x) = \frac{1}{2}(25 - x^2)^{\left(\frac{1}{2} - 1\right)} \frac{d(25 - x^2)}{dx}$$

$$\Rightarrow f'(x) = \frac{1}{2}(25 - x^2)^{-\frac{1}{2}}(-2x)$$

$$\Rightarrow f'(x) = \frac{-2x}{2\,(25 - x^2)^{\frac{1}{2}}}$$

$$\Rightarrow f'(x) = \frac{-2x}{2\,(25 - x^2)^{\frac{1}{2}}}$$

$$\Rightarrow f'(x) = \frac{-x}{\sqrt{25 - x^2}}$$

Here also,

$\sqrt{25 - x^2} > 0$

$\Rightarrow -5 < x < 5$

$\therefore$ f (x) is differentiable in $(-3, 4)$

So both the necessary conditions of Lagrange's mean value theorem is satisfied. Therefore, there exist a point $c \in (-3, 4)$ such that:

$$f'(c) = \frac{f(4) - f(-3)}{4 - (-3)}$$

$$\Rightarrow f'(c)=\frac{f(4)-f(-3)}{4+3}$$

$$\Rightarrow f'(c)=\frac{f(4)-f(-3)}{7}$$

$$f(x)=\left(25-x^2\right)^{\frac{1}{2}}$$

On differentiating with respect to x:

$$f'(x)=\frac{-x}{\sqrt{25-x^2}}$$

For f′(c), put the value of x = c in f′(x):

$$f'(c)=\frac{-c}{\sqrt{25-c^2}}$$

For f (4), put the value of x = 4 in f(x):

$$f(4)=\left(25-4^2\right)^{\frac{1}{2}}$$

$$\Rightarrow f(4)=(25-16)^{\frac{1}{2}}$$

$$\Rightarrow f(4)=(9)^{\frac{1}{2}}$$

$\Rightarrow$ f (4) = 3

For f (– 3), put the value of x = – 3 in f(x):

$$f(-3)=\left(25-(-3)^2\right)^{\frac{1}{2}}$$

$$\Rightarrow f(-3)=(25-9)^{\frac{1}{2}}$$

$$\Rightarrow f(-3)=(16)^{\frac{1}{2}}$$

$\Rightarrow$ f (– 3) = 4

$$f'(c)=\frac{f(4)-f(-3)}{7}$$

$$\Rightarrow \frac{-c}{\sqrt{25-c^2}} = \frac{3-4}{7}$$

$$\Rightarrow \frac{-c}{\sqrt{25-c^2}} = \frac{-1}{7}$$

$$\Rightarrow -7c = -\sqrt{25-c^2}$$

Squaring on both sides:

$$\Rightarrow (-7c)^2 = (-\sqrt{25-c^2})^2$$

$$\Rightarrow 49c^2 = 25 - c^2$$

$$\Rightarrow 50c^2 = 25$$

$$\Rightarrow c^2 = \frac{25}{50}$$

$$\Rightarrow c^2 = \frac{1}{2}$$

$$\Rightarrow c = \pm\frac{1}{\sqrt{2}} \in (-3, 4)$$

Hence, Lagrange's mean value theorem is verified.

(x) f (x) = $\tan^{-1}x$ on [0, 1]

Solution:

Given f (x) = $\tan^{-1} x$ on [0, 1]

$\text{Tan}^{-1} x$ has unique value for all x between 0 and 1.

∴ f (x) is continuous in [0, 1]

f (x) = $\tan^{-1} x$

Differentiating with respect to x:

$$f'(x) = \frac{1}{1+x^2}$$

x^2 always has value greater than 0.

$\Rightarrow 1 + x^2 > 0$

∴ f (x) is differentiable in (0, 1)

So both the necessary conditions of Lagrange's mean value theorem is satisfied.
Therefore, there exist a point c ∈ (0, 1) such that:

$$f'(c) = \frac{f(1) - f(0)}{1 - 0}$$

$\Rightarrow f'(c) = f(1) - f(0)$

$f(x) = \tan^{-1} x$

Differentiating with respect to x:

$$f'(x) = \frac{1}{1+x^2}$$

For f′(c), put the value of x=c in f′(x):

$$f'(c) = \frac{1}{1+c^2}$$

For f (1), put the value of x=1 in f(x):

$f(1) = \tan^{-1} 1$

$$\Rightarrow f(1) = \frac{\pi}{4}$$

For f (0), put the value of x=0 in f(x):

$f(0) = \tan^{-1} 0$

$\Rightarrow f(0) = 0$

$f'(c) = f(1) - f(0)$

$$\Rightarrow \frac{1}{1+c^2} = \frac{\pi}{4} - 0$$

$$\Rightarrow \frac{1}{1+c^2} = \frac{\pi}{4}$$

$\Rightarrow 4 = \pi(1+c^2)$

$\Rightarrow 4 = \pi + \pi c^2$

$\Rightarrow -\pi c^2 = \pi - 4$

$$\Rightarrow c^2 = \frac{\pi - 4}{-\pi}$$

$$\Rightarrow c^2 = \frac{4 - \pi}{\pi}$$

$$\Rightarrow c = \sqrt{\frac{4}{\pi} - 1} \approx 0.52 \in (0, 1)$$

Hence, Lagrange's mean value theorem is verified.

(xi) $f(x) = x + \frac{1}{x}$ **on** $[1, 3]$

Solution:

Given

$$f(x) = x + \frac{1}{x} \text{ on } [1, 3]$$

F (x) has unique values for all $x \in (1, 3)$

∴ f (x) is continuous in [1, 3]

$$f(x) = x + \frac{1}{x} \text{ on } [1, 3]$$

Differentiating with respect to x

$$f'(x) = 1 + (-1)(x)^{-2}$$

$$\Rightarrow f'(x) = 1 - \frac{1}{x^2}$$

$$\Rightarrow f'(x) = \frac{x^2 - 1}{x^2}$$

Here, $x^2 \neq 0$

⇒ f'(x) exists for all values except 0

∴ f (x) is differentiable in (1, 3)

So both the necessary conditions of Lagrange's mean value theorem is satisfied. Therefore, there exist a point c ∈ (1, 3) such that:

$$f'(c)=\frac{f(3)\ -\ f(1)}{3-1}$$

$$\Rightarrow f'(c)=\frac{f(3)-f(1)}{2}$$

$$f(x)=x+\frac{1}{x}$$

On differentiating with respect to x:

$$f'(x)=\frac{x^2-1}{x^2}$$

For f'(c), put the value of x=c in f'(x):

$$f'(c)=\frac{c^2-1}{c^2}$$

For f (3), put the value of x = 3 in f(x):

$$f(3)=3+\frac{1}{3}$$

$$\Rightarrow f(3)=\frac{9+1}{3}$$

$$\Rightarrow f(3)=\frac{10}{3}$$

For f (1), put the value of x = 1 in f(x):

$$f(1)=1+\frac{1}{1}$$

$$\Rightarrow f(1)=2$$

$$\Rightarrow f'(c)=\frac{f(3)\ -\ f(1)}{2}$$

$$\Rightarrow \frac{c^2-1}{c^2}=\frac{\frac{10}{3}-2}{2}$$

$\Rightarrow 2(c^2 - 1) = c^2\left(\frac{10}{3} - 2\right)$

$\Rightarrow 2(c^2 - 1) = c^2\left(\frac{10 - 6}{3}\right)$

$\Rightarrow 2(c^2 - 1) = c^2\left(\frac{4}{3}\right)$

$\Rightarrow 6(c^2 - 1) = 4c^2$

$\Rightarrow 6c^2 - 6 = 4c^2$

$\Rightarrow 6c^2 - 4c^2 = 6$

$\Rightarrow 2c^2 = 6$

$\Rightarrow c^2 = \frac{6}{2}$

$\Rightarrow c^2 = 3$

$\Rightarrow c = \pm\sqrt{3} \in (-3, 4)$

Hence, Lagrange's mean value theorem is verified.

(xii) f (x) = x (x + 4)^2 on [0, 4]

Solution:
Given f (x) = x (x + 4)^2 on [0, 4]
= x [(x)^2 + 2 (4) (x) + (4)^2]
= x (x^2 + 8x + 16)
= x^3 + 8x^2 + 16x on [0, 4]
Every polynomial function is continuous everywhere on (−∞, ∞) and differentiable for all arguments. Here, f(x) is a polynomial function. So it is continuous in [0, 4] and differentiable in (0, 4). So both the necessary conditions of Lagrange's mean value theorem is satisfied. Therefore, there exist a point c ∈ (0, 4) such that:

$f'(c) = \frac{f(4) - f(0)}{4 - 0}$

$\Rightarrow f'(c) = \frac{f(4) - f(0)}{4}$

$f(x) = x^3 + 8x^2 + 16x$

Differentiating with respect to x:

$f'(x) = 3x^2 + 8(2x) + 16$

$= 3x^2 + 16x + 16$

For f'(c), put the value of x = c in f'(x):

$f'(c) = 3c^2 + 16c + 16$

For f (4), put the value of x = 4 in f(x):

$f(4) = (4)^3 + 8(4)^2 + 16(4)$

$= 64 + 128 + 64$

$= 256$

For f (0), put the value of x = 0 in f(x):

$f(0) = (0)^3 + 8(0)^2 + 16(0)$

$= 0 + 0 + 0$

$= 0$

$$f'(c) = \frac{f(4) - f(0)}{4}$$

$$\Rightarrow 3c^2 + 16c + 16 = \frac{256 - 0}{4}$$

$$\Rightarrow 3c^2 + 16c + 16 = \frac{256}{4}$$

$$\Rightarrow 3c^2 + 16c + 16 = 64$$

$$\Rightarrow 3c^2 + 16c + 16 - 64 = 0$$

$$\Rightarrow 3c^2 + 16c - 48 = 0$$

For quadratic equation, $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow c = \frac{-(16) \pm \sqrt{(16)^2 - 4 \times 3 \times (-48)}}{2 \times 3}$$

$$\Rightarrow c = \frac{-16 \pm \sqrt{256 + 576}}{6}$$

$$\Rightarrow c = \frac{-16 \pm \sqrt{832}}{6}$$

$$\Rightarrow c = \frac{-16 \pm 8\sqrt{13}}{6}$$

$$\Rightarrow c = \frac{-16}{6} \pm \frac{8\sqrt{13}}{6}$$

$$\Rightarrow c = \frac{-8}{3} \pm \frac{4\sqrt{13}}{3}$$

$$\Rightarrow c = \frac{-8}{3} + \frac{4\sqrt{13}}{3}, \frac{-8}{3} - \frac{4\sqrt{13}}{3} \in c$$

Hence, Lagrange's mean value theorem is verified.

(xiii) $f(x) = \sqrt{x^2 - 4}$ **on [2, 4]**

Solution:

Given

$f(x) = \sqrt{x^2 - 4}$ on [2, 4]

Here,

$\sqrt{x^2 - 4} > 0$

$\Rightarrow x^2 - 4 > 0$

$\Rightarrow x^2 > 4$

$\Rightarrow$ f (x) exists for all values expect (– 2, 2)

$\therefore$ f (x) is continuous in [2, 4]

$f(x) = \sqrt{x^2 - 4}$

Differentiating with respect to x:

$$f'(x) = \frac{1}{2}(x^2 - 4)^{\left(\frac{1}{2} - 1\right)} \frac{d(x^2 - 4)}{dx}$$

$\Rightarrow f'(x)=\frac{1}{2}(x^2-4)^{-\frac{1}{2}}(2x)$

$\Rightarrow f'(x)=\frac{2x}{2(x^2-4)^{\frac{1}{2}}}$

$\Rightarrow f'(x)=\frac{x}{\sqrt{x^2-4}}$

Here also, $\sqrt{x^2-4}>0$

$\Rightarrow$ f′(x) exists for all values of x except (2, – 2)

∴ f (x) is differentiable in (2, 4)

So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exist a point c ∈ (2, 4) such that:

$f'(c)=\frac{f(4)-f(2)}{4-2}$

$\Rightarrow f'(c)=\frac{f(4)-f(2)}{2}$

$f(x)=\sqrt{x^2-4}$

On differentiating with respect to x:

$f'(x)=\frac{x}{\sqrt{x^2-4}}$

For f′(c), put the value of x=c in f′(x):

$f'(c)=\frac{c}{\sqrt{c^2-4}}$

For f (4), put the value of x = 4 in f(x):

$f(4)=\sqrt{4^2-4}$

$\Rightarrow f(4) = (16-4)^{\frac{1}{2}}$

$\Rightarrow f(4) = \sqrt{12}$

$\Rightarrow f(4) = 2\sqrt{3}$

For f (2), put the value of x = 2 in f(x):

$f(2) = \sqrt{2^2 - 4}$

$\Rightarrow f(2) = (4-4)^{\frac{1}{2}}$

$\Rightarrow f(2) = 0$

$\Rightarrow f'(c) = \frac{f(4) - f(2)}{2}$

$\Rightarrow \frac{c}{\sqrt{c^2 - 4}} = \frac{2\sqrt{3} - 0}{2}$

$\Rightarrow \frac{c}{\sqrt{c^2 - 4}} = \sqrt{3}$

$\Rightarrow c = (\sqrt{3})\sqrt{c^2 - 4}$

Squaring both sides:

$\Rightarrow (c)^2 = ((\sqrt{3})\sqrt{c^2 - 4})^2$

$\Rightarrow c^2 = 3(c^2 - 4)$

$\Rightarrow c^2 = 3c^2 - 12$

$\Rightarrow -2c^2 = -12$

$\Rightarrow c^2 = \frac{-12}{-2}$

$\Rightarrow c^2 = 6$

$\Rightarrow c = \pm\sqrt{6}$

$\Rightarrow c = \sqrt{6} \in (2, 4)$

Hence, Lagrange's mean value theorem is verified.

(xiv) f (x) = x^2 + x – 1 on [0, 4]

Solution:
Given f (x) = $x^2 + x - 1$ on [0, 4]
Every polynomial function is continuous everywhere on (−∞, ∞) and differentiable for all arguments. Here, f(x) is a polynomial function. So it is continuous in [0, 4] and differentiable in (0, 4). So both the necessary conditions of Lagrange's mean value theorem is satisfied. Therefore, there exist a point c ∈ (0, 4) such that:

$$f'(c) = \frac{f(4) - f(0)}{4 - 0}$$

$$\Rightarrow f'(c) = \frac{f(4) - f(0)}{4}$$

f (x) = $x^2 + x - 1$
Differentiating with respect to x:
f'(x) = 2x + 1
For f'(c), put the value of x = c in f'(x):
f'(c) = 2c + 1
For f (4), put the value of x = 4 in f(x):
f (4)= $(4)^2 + 4 - 1$
= 16 + 4 – 1
= 19
For f (0), put the value of x = 0 in f(x):
f (0) = $(0)^2 + 0 - 1$
= 0 + 0 – 1
= – 1

$$f'(c) = \frac{f(4) - f(0)}{4}$$

$$\Rightarrow 2c + 1 = \frac{19 - (-1)}{4}$$

$$\Rightarrow 2c + 1 = \frac{20}{4}$$

$\Rightarrow 2c + 1 = 5$

$\Rightarrow 2c = 5 - 1$

$\Rightarrow 2c = 4$

$\Rightarrow c = \frac{4}{2} = 2 \in (0, 4)$

Hence, Lagrange's mean value theorem is verified.

(xv) f (x) = sin x – sin 2x – x on [0, π]

Solution:
Given f (x) = sin x – sin 2x – x on [0, π]
Sin x and cos x functions are continuous everywhere on (−∞, ∞) and differentiable for all arguments. So both the necessary conditions of Lagrange's mean value theorem is satisfied. Therefore, there exist a point c ∈ (0, π) such that:

$f'(c) = \frac{f(\pi) - f(0)}{\pi - 0}$

$\Rightarrow f'(c) = \frac{f(\pi) - f(0)}{\pi}$

f (x) = sin x – sin 2x – x

Differentiating with respect to x:

f (x) = sin x – sin 2x – x

$\Rightarrow f'(x) = \cos x - \cos 2x \frac{d(2x)}{dx} - 1$

$\Rightarrow f'(x) = \cos x - 2\cos 2x - 1$

For f'(c), put the value of x=c in f'(x):

f'(c) = cos c – 2cos 2c – 1

For f (π), put the value of x = π in f(x):

f (π) = sin π – sin 2π – π

= 0 – 0 – π

$= -\pi$

For f (0), put the value of x=0 in f(x):

$f(0) = \sin 0 - \sin 2(0) - 0$

$= \sin 0 - \sin 0 - 0$

$= 0 - 0 - 0$

$= 0$

$$f'(c) = \frac{f(\pi) - f(0)}{\pi}$$

$$\Rightarrow \cos c - 2\cos 2c - 1 = \frac{-\pi - 0}{\pi}$$

$\Rightarrow \text{Cos } c - 2\cos 2c - 1 = -1$

$\Rightarrow \text{Cos } c - 2(2\cos^2 c - 1) = -1 + 1$

$\Rightarrow \text{Cos } c - 4\cos^2 c + 2 = 0$

$\Rightarrow 4\cos^2 c - \cos c - 2 = 0$

For quadratic equation, $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \cos c = \frac{-(-1) \pm \sqrt{(-1)^2 - 4\times 4\times(-2)}}{2\times 4}$$

$$\Rightarrow \cos c = \frac{1 \pm \sqrt{1 + 32}}{8}$$

$$\Rightarrow \cos c = \frac{1 \pm \sqrt{33}}{8}$$

$$\Rightarrow c = \cos^{-1}\left(\frac{1 \pm \sqrt{33}}{8}\right) \in (0, \pi)$$

Hence, Lagrange's mean value theorem is verified.

(xvi) f (x) = $x^3 - 5x^2 - 3x$ on [1, 3]

Solution:
Given f (x) = $x^3 - 5x^2 - 3x$ on [1, 3]
Every polynomial function is continuous everywhere on (−∞, ∞) and differentiable for all arguments. Here, f(x) is a polynomial function. So it is continuous in [1, 3] and differentiable in (1, 3). So both the necessary conditions of Lagrange's mean value theorem is satisfied.
Therefore, there exist a point c ∈ (1, 3) such that:

$$f'(c)=\frac{f(3)-f(1)}{3-1}$$

$$\Rightarrow f'(c)=\frac{f(3)-f(1)}{2}$$

f (x) = $x^3 - 5x^2 - 3x$
Differentiating with respect to x:
f'(x) = $3x^2 - 5(2x) - 3$
= $3x^2 - 10x - 3$
For f'(c), put the value of x=c in f'(x):
f'(c)= $3c^2 - 10c - 3$
For f (3), put the value of x = 3 in f(x):
f (3)= $(3)^3 - 5(3)^2 - 3(3)$
= 27 – 45 – 9
= – 27
For f (1), put the value of x = 1 in f(x):
f (1)= $(1)^3 - 5(1)^2 - 3(1)$
= 1 – 5 – 3
= – 7

$$f'(c)=\frac{f(3)-f(1)}{2}$$

$$\Rightarrow 3c^2-10c-3=\frac{(-27)-(-7)}{2}$$

$$\Rightarrow 3c^2-10c-3=\frac{-27+7}{2}$$

$$\Rightarrow 3c^2-10c-3=\frac{-20}{2}$$

$\Rightarrow 3c^2 - 10c - 3 = -10$
$\Rightarrow 3c^2 - 10c - 3 + 10 = 0$
$\Rightarrow 3c^2 - 10c + 7 = 0$
$\Rightarrow 3c^2 - 7c - 3c + 7 = 0$
$\Rightarrow c(3c - 7) - 1(3c - 7) = 0$
$\Rightarrow (3c - 7)(c - 1) = 0$

$$\Rightarrow c = \frac{7}{3}, 1$$

$$\Rightarrow c = \frac{7}{3} \in (1, 3)$$

Hence, Lagrange's mean value theorem is verified.

2. Discuss the applicability of Lagrange's mean value theorem for the function f(x) = |x| on [– 1, 1].

Solution:

Given f (x) = |x| on [– 1, 1]

$$\text{So } f(x) \text{ can be defined as} = \begin{cases} -x, & x<0 \\ x, & x\geq 0 \end{cases}$$

For differentiability at x = 0,

$$\text{LHD} = \lim_{x\to 0^-} \frac{f(0-h) - f(0)}{-h}$$

{Since f(x) = – x, x<0}

$$= \lim_{x\to 0^-} \frac{-(0-h) - 0}{-h}$$

$$= \lim_{x\to 0^-} \frac{h-0}{-h}$$

$$= \lim_{x\to 0^-} \frac{h}{-h}$$

$= -1$

$$\text{RHD} = \lim_{x\to 0^+} \frac{f(0-h) - f(0)}{-h}$$

{Since f(x) = x, x>0}

$$= \lim_{x \to 0^-} \frac{(0-h)-0}{-h}$$

$$= \lim_{x \to 0^-} \frac{-h-0}{-h}$$

$$= \lim_{x \to 0^-} \frac{-h}{-h}$$

= 1

LHD ≠ RHD

⇒ f (x) is not differential at x=0

∴ Lagrange's mean value theorem is not applicable for the function f(x) = |x| on [– 1, 1].

3. Show that the Lagrange's mean value theorem is not applicable to the function f(x) = 1/x on [–1, 1].

Solution:

Given $f(x) = \frac{1}{x}$ on $[-1, 1]$

Here, x ≠ 0

⇒ f (x) exists for all values of x except 0

⇒ f (x) is discontinuous at x=0

∴ f (x) is not continuous in [– 1, 1]

Hence the Lagrange's mean value theorem is not applicable to the function f (x) = 1/x on [-1, 1]

4. Verify the hypothesis and conclusion of Lagrange's mean value theorem for the function

$$f(x) = \frac{1}{4x-1}, 1 \le x \le 4.$$

Solution:

Given

$f(x) = \frac{1}{4x - 1}$ on [1, 4]

Where 4x – 1>0

f'(x) has unique values for all x except ¼

∴ f (x) is continuous in [1, 4]

$f(x) = \frac{1}{4x - 1}$

Differentiating with respect to x:

$f'(x) = (-1)(4x - 1)^{-2}(4)$

$\Rightarrow f'(x) = -\frac{4}{(4x - 1)^2}$

Here, 4x – 1>0

f'(x) has unique values for all x except ¼

∴ f (x) is differentiable in (1, 4)

So both the necessary conditions of Lagrange's mean value theorem is satisfied. Therefore, there exist a point c ∈ (1, 4) such that:

$f'(c) = \frac{f(4) - f(1)}{4 - 1}$

$\Rightarrow f'(c) = \frac{f(4) - f(1)}{3}$

$f(x) = \frac{1}{4x - 1}$

On differentiating with respect to x:

$f'(x) = -\frac{4}{(4x - 1)^2}$

For f'(c), put the value of x=c in f'(x):

$$f'(c) = -\frac{4}{(4c-1)^2}$$

For f (4), put the value of x = 4 in f (x):

$$f(4) = \frac{1}{4(4)-1}$$

$$\Rightarrow f(4) = \frac{1}{16-1}$$

$$\Rightarrow f(4) = \frac{1}{15}$$

For f (1), put the value of x = 1 in f(x):

$$f(1) = \frac{1}{4(1)-1}$$

$$\Rightarrow f(1) = \frac{1}{4-1}$$

$$\Rightarrow f(1) = \frac{1}{3}$$

$$\Rightarrow f'(c) = \frac{f(4)-f(1)}{3}$$

$$\Rightarrow -\frac{4}{(4c-1)^2} = \frac{\frac{1}{15}-\frac{1}{3}}{3}$$

$$\Rightarrow -3(4) = (4c-1)^2\left(\frac{1}{15}-\frac{1}{3}\right)$$

$$\Rightarrow -12 = (4c-1)^2\left(\frac{3-15}{45}\right)$$

$$\Rightarrow -12 = (4c-1)^2\left(\frac{-12}{45}\right)$$

$$\Rightarrow -12 \times \frac{45}{-12} = (4c-1)^2$$

$$\Rightarrow -12\times\frac{45}{-12}=(4c-1)^2$$

$$\Rightarrow (4c-1)^2 = 45$$

$$\Rightarrow (4c-1) = \pm\sqrt{45}$$

$$\Rightarrow (4c-1) = \pm 3\sqrt{5}$$

$$\Rightarrow c = \frac{\pm 3\sqrt{5}+1}{4}$$

$$\Rightarrow c = \frac{3\sqrt{5}+1}{4} \approx 1.92 \in (1, 4)$$

Hence, Lagrange's mean value theorem is verified.

5. Find a point on the parabola $y = (x - 4)^2$, where the tangent is parallel to the chord joining (4, 0) and (5, 1).

Solution:

Given $f(x) = (x - 4)^2$ on [4, 5]

This interval [a, b] is obtained by x – coordinates of the points of the chord.

Every polynomial function is continuous everywhere on $(-\infty, \infty)$ and differentiable for all arguments. Here, f(x) is a polynomial function. So it is continuous in [4, 5] and differentiable in (4, 5). So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exist a point $c \in (4, 5)$ such that:

$$f'(c)=\frac{f(5)-f(4)}{5-4}$$

$$\Rightarrow f'(c)=\frac{f(5)-f(4)}{1}$$

$$f(x) = (x-4)^2$$

Differentiating with respect to x:

$$f'(x)=2(x-4)\frac{d(x-4)}{dx}$$

$$\Rightarrow f'(x) = 2(x-4)(1)$$

⇒ $f'(x) = 2(x - 4)$

For f'(c), put the value of x=c in f'(x):

$f'(c) = 2(c - 4)$

For f (5), put the value of x=5 in f(x):

$f(5) = (5 - 4)^2$

$= (1)^2$

$= 1$

For f (4), put the value of x=4 in f(x):

$f(4) = (4 - 4)^2$

$= (0)^2$

$= 0$

$f'(c) = f(5) - f(4)$

$\Rightarrow 2(c - 4) = 1 - 0$

$\Rightarrow 2c - 8 = 1$

$\Rightarrow 2c = 1 + 8$

$\Rightarrow c = \frac{9}{2} = 4.5 \in (4, 5)$

We know that, the value of c obtained in Lagrange's Mean value Theorem is nothing but the value of x – coordinate of the point of the contact of the tangent to the curve which is parallel to the chord joining the points (4, 0) and (5, 1).

Now, put this value of x in f(x) to obtain y:

$y = (x - 4)^2$

$$\Rightarrow y = \left(\frac{9}{2} - 4\right)^2$$

$$\Rightarrow y = \left(\frac{9 - 8}{2}\right)^2$$

$$\Rightarrow y = \left(\frac{1}{2}\right)^2$$

$$\Rightarrow y = \frac{1}{4}$$

Hence, the required point is $\left(\frac{9}{2}, \frac{1}{4}\right)$