EXERCISE 10.1 PAGE NO: 10.10

1. Show that f (x) = |x – 3| is continuous but not differentiable at x = 3.

Solution:

Given f (x) = |x – 3|

Therefore we can write given function as,

$$f(x) = \begin{cases} -(x-3), x < 3 \\ x-3, x \geq 3 \end{cases}$$

But f (3) = 3 - 3 = 0

LHL = $\lim_{x \to 3} f(x)$

= $\lim_{h \to 0} f(3-h)$

= $\lim_{h \to 0} 3-(3-h)$

= $\lim_{h \to 0} 0$

Now consider,

RHL = $\lim_{x \to 3} f(x)$

= $\lim_{h \to 0} f(3 + h)$

= $\lim_{h \to 0} 3 + h - 3$

= 0

LHL = RHL = f (3)

Since, f(x) is continuous at x = 3

LHD at x = 3 = $\lim_{x-3^-} \frac{f(x)-f(3)}{x-3}$

= $\lim_{h-0^-} \frac{f(3-h)-f(3)}{3-h-3}$

= $\lim_{h-0^-} \frac{3-(3-h)-0}{-h}$

= $\lim_{h-0^-} \frac{h}{-h}$

= - 1

RHD at x = 3 = $\lim_{x-3^+} \frac{f(x)-f(3)}{x-3}$

= $\lim_{h-0^+} \frac{f(3+h)-f(3)}{3+h-3}$

= $\lim_{h-0^+} \frac{3+h-3-0}{h}$

= $\lim_{h-0^+} \frac{h}{h}$

= 1

LHD at x = 3 ≠ RHD at x = 3

Hence, f(x) is continuous but not differentiable at x = 3.

2. Show that f (x) = x $^{1/3}$ is not differentiable at x = 0.

Solution:

For differentiability,

LHD (at x = 0) = RHD (at x = 0)

(LHD at x = 0) = $\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0}$

= $\lim_{h\to 0^-} \frac{f(0-h)-f(0)}{0-h-0}$

$= \lim_{h\to 0^-} \frac{(-h)^{\frac{1}{3}}-0}{-h}$

$= \lim_{h\to 0^-} \frac{(-h)^{\frac{1}{3}}}{-h}$

$= \lim_{h\to 0^-} \frac{(-1)^{\frac{1}{3}}(h)^{\frac{1}{3}}}{(-1)h}$

$= \lim_{h\to 0}(-1)^{\frac{-2}{3}}\ h^{\frac{-2}{3}}$

= Not defined

(RHD at x = 3) $= \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0}$

$= \lim_{h\to 0^-} \frac{f(0+h)-f(0)}{0+h-0}$

$= \lim_{h\to 0^+} \frac{(h)^{\frac{1}{3}}-0}{+h}$

$= \lim_{h\to 0^+} \frac{(h)^{\frac{1}{3}}}{+h}$

$= \lim_{h\to 0} h^{\frac{-2}{3}}$

= Not defined

Since, LHD and RHD does not exist at x = 0

Hence, f(x) is not differentiable at x = 0

3. *Show that* $f(x) = \begin{cases} 12x-13, \ if\ x \le 3 \\ 2x^2+5, \ if\ x > 3 \end{cases}$ *is differentiable at* $x = 3$. *Also, find* $f'(3)$

Solution:

Now we have to check differentiability of given function at x = 3

That is LHD (at x = 3) = RHD (at x = 3)

(LHD at x = 3) = $\lim_{x-3^-} \frac{f(x)-f(3)}{x-3}$

$= \lim_{h-0^-} \frac{f(3-h)-f(3)}{3-h-3}$

$= \lim_{h-0^-} \frac{[12(3-h)-13]-[12(3)-13]}{-h}$

$= \lim_{h-0^-} \frac{36-12h-13-36+13}{-h}$

$= \lim_{h-0^-} \frac{-12h}{-h}$

= 12

(RHD at x = 3) = $\lim_{x-3^+} \frac{f(x)-f(3)}{x-3}$

$= \lim_{h\to 0^+} \frac{f(3+h)-f(3)}{3+h-3}$

$= \lim_{h\to 0^+} \frac{[2(3+h^2)+5]-[12(3)-13]}{3+h-3}$

$= \lim_{h\to 0^+} \frac{18+12h+2h^2+5-36+13}{h}$

$= \lim_{h\to 0^+} \frac{2h^2+12h}{h}$

$= \lim_{h\to 0^+} \frac{h(2h+12)}{h}$

= 12

Since, (LHD at x = 3) = (RHD at x = 3)

Hence, f(x) is differentiable at x = 3.

4. Show that the function f is defined as follows

$$f(x) = \begin{cases} 3x-2, \; 0 < x \le 1 \\ 2x^2 - x, \; 1 < x \le 2 \\ 5x-4, \; x > 2 \end{cases}$$

Is continuous at x = 2, but not differentiable thereat.

Solution:

Given

$$f(x) = \begin{cases} 3x - 2,\ 0 < x \le 1 \\ 2x^2 - x,\ 1 < x \le 2 \\ 5x - 4,\ x > 2 \end{cases}$$

Now we have to check continuity at x = 2

For continuity,

LHL (at x = 2) = RHL (at x = 2)

f (2) = $2(2)^2 - 2$

= 8 - 2 = 6

LHL = $\lim_{x\to 2^-} f(x)$

= $\lim_{h\to 0^-} f(2-h)$

= $\lim_{h\to 0^-} [2(2-h)^2 - (2-h)]$

= 8 - 2

= 6

RHL = $\lim_{x\to 2^+} f(x)$

= $\lim_{h\to 0^+} f(2 + h)$

= $\lim_{h\to 0} 5(2 + h) - 4$

= 6

Since, LHL = RHL = f (2)

Hence, F(x) is continuous at x = 2

Now we have to differentiability at x = 2

(LHD at x = 2) = $\lim_{x \to 2^-} \frac{f(x)-f(2)}{x-2}$

$= \lim_{h \to 0} \frac{f(2-h)-f(2)}{2-h-2}$

$= \lim_{h \to 0} \frac{[2(2-h)^2-(2-h)]-[8-2]}{-h}$

$= \lim_{h \to 0} \frac{8-8h+2h^2-h-6}{-h}$

$= \lim_{h \to 0} \frac{2h^2-6h}{-h}$

$= \lim_{h \to 0} \frac{h(2h-6)}{-h}$

$= \lim_{h \to 0} (6-2h)$

= 6

Now consider,

(RHD at x = 2) = $\lim_{x \to 2^+} \frac{f(x)-f(2)}{x-2}$

$= \lim_{h \to 0} \frac{f(2+h)-f(2)}{2+h-2}$

$= \lim_{h \to 0} \frac{[5(2+h)-4]-[8-2]}{h}$

$= \lim_{h \to 0} \frac{10+5h-4-6}{h}$

= 5

Since, (RHD at x = 2) ≠ (LHD at x = 2)

Hence, f (2) is not differentiable at x = 2.

5. Discuss the continuity and differentiability of the function f (x) = |x| + |x -1| in the interval of (-1, 2).

Solution:

The given function f (x) can be defined as

f (x) = $\begin{cases} x + x + 1, -1 < x < 0 \\ 1, 0 \le x \le 1 \\ -x - x + 1, 1 < x < 2 \end{cases}$

f (x) = $\begin{cases} 2x + 1, -1 < x < 0 \\ 1, 0 \le x \le 1 \\ -2x + 1, 1 < x < 2 \end{cases}$

We know that a polynomial and a constant function is continuous and differentiable everywhere. So, f(x) is continuous and differentiable for x $\in$
(-1, 0) and x ∈ (0, 1) and (1, 2).
We need to check continuity and differentiability at x = 0 and x = 1.
Continuity at x = 0

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 2x + 1$ = 1

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 1$ = 1

F (0) = 1

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$

Since, f(x) is continuous at x = 0

Continuity at x = 1

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 1$ = 1

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 1$ = 1

F (1) = 1

$\log_{x\to 0^+} f(x) = \log_{x\to 0^-} f(x)$ = 1

Since, f(x) is continuous at x = 1
Now we have to check differentiability at x = 0
For differentiability, LHD (at x = 0) = RHD (at x = 0)
Differentiability at x = 0

(LHD at x = 0) = $\lim_{x \to 0^-} \frac{f(x)-f(0)}{x-0}$

= $\lim_{x \to 0^-} \frac{2x+1-1}{x-0}$

= $\lim_{x \to 0^-} \frac{2x}{x}$

= 2

(RHD at x = 0) = $\lim_{x \to 0^+} \frac{f(x)-f(0)}{x-0}$

= $\lim_{x \to 0^+} \frac{1-1}{x}$

= $\lim_{x \to 0^+} \frac{0}{x}$

= 0

Since, (LHD at x = 0) ≠ (RHD at x = 0)
So, f(x) is differentiable at x = 0.
Now we have to check differentiability at x = 1
For differentiability, LHD (at x = 1) = RHD (at x = 1)
Differentiability at x = 1

(LHD at x = 1) = $\lim_{x \to 1^-} \frac{f(x)-f(1)}{x-1}$

= $\lim_{x \to 1^-} \frac{1-1}{x-1}$

= 0

(RHD at x = 1) = $\lim_{x \to 1^+} \frac{f(x)-f(1)}{x-1}$

= $\lim_{x \to 1^+} \frac{-2x+1-1}{x-1}$

= ∞

Since, f(x) is not differentiable at x = 1.
So, f(x) is continuous on (- 1, 2) but not differentiable at x = 0, 1

EXERCISE 10.2 PAGE NO: 10.16

1. If f is defined by f (x) = x^2, find f' (2).

Solution:

We have a polynomial function f(x) = x^2, and we have to find whether it is derivable at x = 2 or not, so by using the formula, f '(c) = $\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$,

We get, f' (2) = $\lim_{x\to 2}\frac{f(x)-f(2)}{x-2}$

f' (2) = $\lim_{x\to 2}\frac{x^2-2^2}{x-2}$

f' (2) = $\lim_{x\to 2}\frac{(x+2)(x-2)}{x-2}$

[Using $a^2 - b^2 = (a + b)(a - b)$]

f' (2) = $\lim_{x\to 2} x + 2 = 4$

Hence, the function is differentiable at x = 2 and its derivative equals to 4.

2. If f is defined by f (x) = $x^2 - 4x + 7$, show that f' (5) = 2 f' (7/2)

Solution:

We have a polynomial function f(x) = $x^2 - 4x + 7$, and we have to f '(x) its value at x = 5 and x = 7/2, so by using the formula, f '(c) = $\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$,

We get, f '(5) = $\lim_{x\to 5}\frac{f(x)-f(5)}{x-5}$

f' (5) = $\lim_{x\to 5}\frac{x^2-4x+7-(5^2-4\times 5+7)}{x-5}$

f' (5) = $\lim_{x\to 5}\frac{x^2-4x-5}{x-5}$

f' (5) = $\lim_{x\to5}\frac{x(x-5)+1(x-5)}{x-5}$

f' (5) = $\lim_{x\to5}$ (x + 1) = 6

Hence to function is differentiable at x = 5 and has value 6.

f' (7/2) = $\lim_{x\to\frac{7}{2}}\frac{f(x)-f(\frac{7}{2})}{x-\frac{7}{2}}$

f' (7/2) = $\lim_{x\to\frac{7}{2}}\frac{x^2-4x+7-[(\frac{7}{2})^2-4\times\frac{7}{2}+7]}{x-\frac{7}{2}}$

f' (7/2) = $\lim_{x\to\frac{7}{2}}\frac{x^2-4x+7-[(\frac{7}{2})^2-4\times\frac{7}{2}+7]}{x-\frac{7}{2}}$

f' (7/2) = $\lim_{x\to\frac{7}{2}}\frac{x^2-4x+\frac{7}{4}}{x-\frac{7}{2}}$

f' (7/2) = $\lim_{x\to\frac{7}{2}}\frac{x^2-4x+\frac{7}{4}}{x-\frac{7}{2}}$

f' (7/2) = $\lim_{x\to\frac{7}{2}}\frac{(2x-1)(2x-7)}{2(2x-7)}$

f' (7/2) = $\lim_{x\to\frac{7}{2}}\frac{(2x-1)}{2}$ = 3

Therefore f' (5) = 2 f' (7/2) = 6,
Hence the proof.

3. Show that the derivative of the function f is given by f (x) = $2x^3 - 9x^2 + 12x + 9$, at x = 1 and x = 2 are equal.

Solution:
We are given with a polynomial function f(x) = $2x^3 - 9x^2 + 12x + 9$, and we have

to find f '(x) at x = 1 and x = 2, so by using the formula, f '(c) $= \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$, we get,

$f'(1) = \lim_{x\to 1} \frac{f(x)-f(1)}{x-1}$

$f'(1) = \lim_{x\to 1} \frac{2x^3-9x^2+12x+9-[2(1)^3-9(1)^2+12(1)+9]}{x-1}$

$f'(1) = \lim_{x\to 1} \frac{2x^3-9x^2+12x-5}{x-1}$

$f'(1) = \lim_{x\to 1} \frac{(x-1)(2x^2-7x+5)}{x-1}$

$f'(1) = \lim_{x\to 1} 2x^2 - 7x + 5 = 0$

For x = 2, we get,

$f'(2) = \lim_{x\to 2} \frac{f(x)-f(2)}{x-2}$

$f'(2) = \lim_{x\to 2} \frac{2x^3-9x^2+12x+9-[2(2)^3-9(2)^2+12(2)+9]}{x-2}$

$f'(2) = \lim_{x\to 2} \frac{2x^3-9x^2+12x-4}{x-2}$

$f'(2) = \lim_{x\to 2} \frac{(x-2)(2x^2-5x+2)}{x-2}$

$f'(2) = \lim_{x\to 2} 2x^2 - 5x + 2 = 0$

Hence they are equal at x = 1 and x = 2.

4. If for the function Ø (x) = λ x^2 + 7x – 4, Ø' (5) = 97, find λ.

Solution:

We have to find the value of λ given in the real function and we are given with the differentiability of the function f(x) = λx^2 + 7x - 4 at x = 5 which is f '(5) = 97, so we will adopt the same process but with a little variation.

So by using the formula, f '(c) $= \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$, we get,

f' (5) $= \lim_{x \to 5} \frac{f(x) - f(5)}{x - 5}$

f' (5) $= \lim_{x \to 5} \frac{\lambda x^2 + 7x - 4 - [\lambda(5)^2 + 7(5) - 4]}{x - 5}$

f' (5) $= \lim_{x \to 5} \frac{\lambda x^2 + 7x - 4 - [\lambda(5)^2 + 7(5) - 4]}{x - 5}$

f' (5) $= \lim_{x \to 5} \frac{\lambda x^2 + 7x - 35 - 25\lambda}{x - 5}$

As the limit has some finite value, then there must be the formation of some indeterminate form like $\frac{0}{0}, \frac{\infty}{\infty}$, so if we put the limit value, then the numerator will also be zero as the denominator, but there must be a factor (x - 5) in the numerator, so that this form disappears.

f' (5) $= \lim_{x \to 5} \frac{(x - 5)(\lambda x + 5\lambda + 7)}{x - 5}$

f' (5) = $\lim_{x \to 5} \lambda x + 5\lambda + 7 = 97$

f' (5) = $10\lambda + 7 = 97$

$10\lambda = 90$

$\lambda = 9$

5. If f (x) = $x^3 + 7x^2 + 8x - 9$, find f' (4).

Solution:
We are given with a polynomial function f(x) = $x^3 + 7x^2 + 8x - 9$, and we have to find whether it is derivable at x = 4 or not,

So by using the formula, f '(c) $= \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$,

We get, f '(4) $= \lim_{x \to 4} \frac{f(x) - f(4)}{x - 4}$

$$f'(4) = \lim_{x\to 4} \frac{x^3 + 7x^2 + 8x - 9 - [4^3 + 7(4)^2 + 8(4) - 9]}{x-4}$$

$$f'(4) = \lim_{x\to 4} \frac{(x-4)(x^2 + 11x + 52)}{x-4}$$

$f'(4) = \lim_{x\to 4} x^2 + 11x + 52$

$f'(4) = 112.$

6. Find the derivative of the function f defined by f (x) = mx + c at x = 0.

Solution:

We are given with a polynomial function f(x) = mx + c, and we have to find whether it is derivable at x = 0 or not,

So by using the formula, $f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$,

We get, $f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$

$$f'(0) = \lim_{x\to 0} \frac{mx + c - [m(0) + c]}{x-0}$$

$$f'(0) = \lim_{x\to 0} \frac{mx + c - c}{x-0}$$

$f'(0) = \lim_{x\to 0} m = m$

This is the derivative of a function at x = 0, and also this is the derivative of this function at every value of x.