EXERCISE 18.1 PAGE NO: 18.11

1. Using binomial theorem, write down the expressions of the following:

(i) $(2x + 3y)^5$

(ii) $(2x - 3y)^4$

(iii) $\left(x - \frac{1}{x}\right)^6$

(iv) $(1 - 3x)^7$

(v) $\left(ax - \frac{b}{x}\right)^6$

(vi) $\left(\sqrt{\frac{x}{a}} - \sqrt{\frac{a}{x}}\right)^6$

(vii) $\left(\sqrt[3]{x} - \sqrt[3]{a}\right)^6$

(viii) $(1 + 2x - 3x^2)^5$

(ix) $\left(x + 1 - \frac{1}{x}\right)^3$

(x) $(1 - 2x + 3x^2)^3$

Solution:

(i) $(2x + 3y)^5$

Let us solve the given expression:

$(2x + 3y)^5 = {}^5C_0 (2x)^5 (3y)^0 + {}^5C_1 (2x)^4 (3y)^1 + {}^5C_2 (2x)^3 (3y)^2 + {}^5C_3 (2x)^2 (3y)^3 + {}^5C_4 (2x)^1 (3y)^4 + {}^5C_5 (2x)^0 (3y)^5$

$= 32x^5 + 5 (16x^4) (3y) + 10 (8x^3) (9y)^2 + 10 (4x)^2 (27y)^3 + 5 (2x) (81y^4) + 243y^5$

$= 32x^5 + 240x^4y + 720x^3y^2 + 1080x^2y^3 + 810xy^4 + 243y^5$

(ii) $(2x - 3y)^4$

Let us solve the given expression:

$(2x - 3y)^4 = {}^4C_0 (2x)^4 (3y)^0 - {}^4C_1 (2x)^3 (3y)^1 + {}^4C_2 (2x)^2 (3y)^2 - {}^4C_3 (2x)^1 (3y)^3 + {}^4C_4 (2x)^0 (3y)^4$

$= 16x^4 - 4 (8x^3) (3y) + 6 (4x^2) (9y^2) - 4 (2x) (27y^3) + 81y^4$

$= 16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4$

(iii) $\left(x - \frac{1}{x}\right)^6$

Let us solve the given expression:

$(x - \frac{1}{x})^6$

$$= {}^6C_0 x^6 \left(\frac{1}{x}\right)^0 - {}^6C_1 x^5 \left(\frac{1}{x}\right)^1 + {}^6C_2 x^4 \left(\frac{1}{x}\right)^2 - {}^6C_3 x^3 \left(\frac{1}{x}\right)^3$$

$$+ {}^6C_4 x^2 \left(\frac{1}{x}\right)^4 - {}^6C_5 x^1 \left(\frac{1}{x}\right)^5 + {}^6C_6 x^0 \left(\frac{1}{x}\right)^6$$

$$= x^6 - 6x^5 \times \frac{1}{x} + 15x^4 \times \frac{1}{x^2} - 20x^3 \times \frac{1}{x^3} + 15x^2 \times \frac{1}{x^4} - 6x \times \frac{1}{x^5} + \frac{1}{x^6}$$

$$= x^6 - 6x^4 + 15x^2 - 20 + \frac{15}{x^2} - \frac{6}{x^4} + \frac{1}{x^6}$$

(iv) $(1 – 3x)^7$

Let us solve the given expression:

$(1 – 3x)^7 = {}^7C_0 (3x)^0 - {}^7C_1 (3x)^1 + {}^7C_2 (3x)^2 - {}^7C_3 (3x)^3 + {}^7C_4 (3x)^4 - {}^7C_6 (3x)^6 - {}^7C_7 (3x)^7$

$= 1 – 7 (3x) + 21 (9x)^2 – 35 (27x^3) + 35 (81x^4) – 21 (243x^5) + 7 (729x^6) – 2187(x^7)$

$= 1 – 21x + 189x^2 – 945x^3 + 2835x^4 – 5103x^5 + 5103x^6 – 2187x^7$

(v) $\left(ax - \frac{b}{x}\right)^6$

Let us solve the given expression:

$$= {}^6C_0 (ax)^6 (\frac{b}{x})^0 - {}^6C_1 (ax)^5 (\frac{b}{x})^1 + {}^6C_2 (ax)^4 (\frac{b}{x})^2 - {}^6C_3 (ax)^3 (\frac{b}{x})^3$$

$$+ {}^6C_4 (ax)^2 (\frac{b}{x})^4 - {}^6C_5 (ax)^1 (\frac{b}{x})^5 + {}^6C_6 (ax)^0 (\frac{b}{x})^6$$

$$= a^6x^6 - 6a^5x^5 \times \frac{b}{x} + 15a^4x^4 \times \frac{b^2}{x^2} - 20a^3b^3 \times \frac{b^3}{x^3} + 15a^2x^2 \times \frac{b^4}{x^4} - 6ax \times \frac{b^5}{x^5} + \frac{b^6}{x^6}$$

$$= a^6x^6 - 6a^5x^4b + 15a^4x^2b^2 - 20a^3b^3 + 15\frac{a^2b^4}{x^2} - 6\frac{ab^5}{x^4} + \frac{b^6}{x^6}$$

(vi) $\left(\sqrt{\frac{x}{a}}-\sqrt{\frac{a}{x}}\right)^6$

Let us solve the given expression:

$$= {}^6C_0\left(\sqrt{\frac{x}{a}}\right)^6\left(\sqrt{\frac{a}{x}}\right)^0 - {}^6C_1\left(\sqrt{\frac{x}{a}}\right)^5\left(\sqrt{\frac{a}{x}}\right)^1 + {}^6C_2\left(\sqrt{\frac{x}{a}}\right)^4\left(\sqrt{\frac{a}{x}}\right)^2 - {}^6C_3\left(\sqrt{\frac{x}{a}}\right)^3\left(\sqrt{\frac{a}{x}}\right)^3$$

$$+ {}^6C_4\left(\sqrt{\frac{x}{a}}\right)^2\left(\sqrt{\frac{a}{x}}\right)^4 - {}^6C_5\left(\sqrt{\frac{x}{a}}\right)^1\left(\sqrt{\frac{a}{x}}\right)^5 + {}^6C_6\left(\sqrt{\frac{x}{a}}\right)^0\left(\sqrt{\frac{a}{x}}\right)^6$$

$$= \frac{x^3}{a^3} - 6\frac{x^2}{a^2} + 15\frac{x}{a} - 20 + 15\frac{a}{x} - 6\frac{a^2}{x^2} + \frac{a^3}{x^3}$$

(vii) $\left(\sqrt[3]{x}-\sqrt[3]{a}\right)^6$

Let us solve the given expression:

$$= {}^6C_0(\sqrt[3]{x})^6(\sqrt[3]{a})^0 - {}^6C_1(\sqrt[3]{x})^5(\sqrt[3]{a})^1 + {}^6C_2(\sqrt[3]{x})^4(\sqrt[3]{a})^2 - {}^6C_3(\sqrt[3]{x})^3(\sqrt[3]{a})^3$$

$$+ {}^6C_4(\sqrt[3]{x})^2(\sqrt[3]{a})^4 - {}^6C_5(\sqrt[3]{x})^1(\sqrt[3]{a})^5 + {}^6C_6(\sqrt[3]{x})^0(\sqrt[3]{a})^6$$

$$= x^2 - 6x^{5/3}a^{1/3} + 15x^{4/3}a^{2/3} - 20xa + 15x^{2/3}a^{4/3} - 6x^{1/3}a^{5/3} + a^2$$

(viii) $(1 + 2x - 3x^2)^5$

Let us solve the given expression:

Let us consider $(1 + 2x)$ and $3x^2$ as two different entities and apply the binomial theorem.

$(1 + 2x - 3x^2)^5 = {}^5C_0 (1 + 2x)^5 (3x^2)^0 - {}^5C_1 (1 + 2x)^4 (3x^2)^1 + {}^5C_2 (1 + 2x)^3 (3x^2)^2 - {}^5C_3 (1 + 2x)^2 (3x^2)^3 + {}^5C_4 (1 + 2x)^1 (3x^2)^4 - {}^5C_5 (1 + 2x)^0 (3x^2)^5$

$= (1 + 2x)^5 - 5(1 + 2x)^4 (3x^2) + 10 (1 + 2x)^3 (9x^4) - 10 (1 + 2x)^2 (27x^6) + 5 (1 + 2x) (81x^8) - 243x^{10}$

$= {}^5C_0 (2x)^0 + {}^5C_1 (2x)^1 + {}^5C_2 (2x)^2 + {}^5C_3 (2x)^3 + {}^5C_4 (2x)^4 + {}^5C_5 (2x)^5 - 15x^2 [{}^4C_0 (2x)^0 + {}^4C_1 (2x)^1 + {}^4C_2 (2x)^2 + {}^4C_3 (2x)^3 + {}^4C_4 (2x)^4] + 90x^4 [1 + 8x^3 + 6x + 12x^2] - 270x^6(1 + 4x^2 + 4x) + 405x^8 + 810x^9 - 243x^{10}$

$= 1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5 - 15x^2 - 120x^3 - 360^4 - 480x^5 - 240x^6 + 90x^4 + 720x^7 + 540x^5 + 1080x^6 - 270x^6 - 1080x^8 - 1080x^7 + 405x^8 + 810x^9 - 243x^{10}$

$= 1 + 10x + 25x^2 - 40x^3 - 190x^4 + 92x^5 + 570x^6 - 360x^7 - 675x^8 + 810x^9 - 243x^{10}$

(ix) $\left(x+1-\frac{1}{x}\right)^3$

Let us solve the given expression:

$$= {}^3C_0(x+1)^3\left(\frac{1}{x}\right)^0 - {}^3C_1(x+1)^2\left(\frac{1}{x}\right)^1 + {}^3C_2(x+1)^1\left(\frac{1}{x}\right)^2 - {}^3C_3(x+1)^0\left(\frac{1}{x}\right)^3$$

$$= (x+1)^3 - 3(x+1)^2 \times \frac{1}{x} + 3\frac{x+1}{x^2} - \frac{1}{x^3}$$

$$= x^3 + 1 + 3x + 3x^2 - \frac{3x^2+3+6x}{x} + 3\frac{x+1}{x^2} - \frac{1}{x^3}$$

$$= x^3 + 1 + 3x + 3x^2 - 3x - \frac{3}{x} - 6 + \frac{3}{x} + \frac{3}{x^2} - \frac{1}{x^3}$$

$$= x^3 + 3x^2 - 5 + \frac{3}{x^2} - \frac{1}{x^3}$$

(x) $(1 - 2x + 3x^2)^3$

Let us solve the given expression:

$$= {}^3C_0(1-2x)^3 + {}^3C_1(1-2x)^2(3x^2) + {}^3C_2(1-2x)(3x^2)^2 + {}^3C_3(3x^2)^3$$

$$= (1-2x)^3 + 9x^2(1-2x)^2 + 27x^4(1-2x) + 27x^6$$

$$= 1 - 8x^3 + 12x^2 - 6x + 9x^2(1 + 4x^2 - 4x) + 27x^4 - 54x^5 + 27x^6$$

$$= 1 - 8x^3 + 12x^2 - 6x + 9x^2 + 36x^4 - 36x^3 + 27x^4 - 54x^5 + 27x^6$$

$$= 1 - 6x + 21x^2 - 44x^3 + 63x^4 - 54x^5 + 27x^6$$

2. Evaluate the following:

(i) $\left(\sqrt{x+1} + \sqrt{x-1}\right)^6 + \left(\sqrt{x+1} - \sqrt{x-1}\right)^6$

(ii) $\left(x + \sqrt{x^2-1}\right)^6 + \left(x - \sqrt{x^2-1}\right)^6$

(iii) $\left(1 + 2\sqrt{x}\right)^5 + \left(1 - 2\sqrt{x}\right)^5$

(iv) $\left(\sqrt{2}+1\right)^6 + \left(\sqrt{2}-1\right)^6$

(v) $\left(3+\sqrt{2}\right)^5-\left(3-\sqrt{2}\right)^5$

(vi) $\left(2+\sqrt{3}\right)^7+\left(2-\sqrt{3}\right)^7$

(vii) $\left(\sqrt{3}+1\right)^5-\left(\sqrt{3}-1\right)^5$

(viii) $(0.99)^5+(1.01)^5$

(ix) $\left(\sqrt{3}+\sqrt{2}\right)^6-\left(\sqrt{3}-\sqrt{2}\right)^6$

(x) $\left\{a^2+\sqrt{a^2-1}\right\}^4+\left\{a^2-\sqrt{a^2-1}\right\}^4$

Solution:

(i) $\left(\sqrt{x+1}+\sqrt{x-1}\right)^6+\left(\sqrt{x+1}-\sqrt{x-1}\right)^6$

Let us solve the given expression:

$= 2[{}^6C_0\,(\sqrt{x+1})^6(\sqrt{x-1})^0+{}^6C_2\,(\sqrt{x+1})^4(\sqrt{x-1})^2$

$+{}^6C_4\,(\sqrt{x+1})^2(\sqrt{x-1})^4+{}^6C_6\,(\sqrt{x+1})^0(\sqrt{x-1})^6]$

$= 2[(x+1)^3+15(x+1)^2(x-1)+15(x+1)(x-1)^2+(x-1)^3$

$= 2[x^3+1+3x+3x^2+15(x^2+2x+1)(x-1)+15(x+1)(x^2+1-2x)+x^3-1+3x-3x^2]$

$= 2[2x^3+6x+15x^3-15x^2+30x^2-30x+15x-15+15x^3+15x^2-30x^2-30x+15x+15]$

$= 2[32x^3-24x]$

$= 16x[4x^2-3]$

(ii) $\left(x+\sqrt{x^2-1}\right)^6+\left(x-\sqrt{x^2-1}\right)^6$

Let us solve the given expression:

$= 2\left[{}^6C_0x^6\left(\sqrt{x^2-1}\right)^0+{}^6C_2x^4\left(\sqrt{x^2-1}\right)^2+{}^6C_4x^2\left(\sqrt{x^2-1}\right)^4+\right.$

$\left.{}^6C_6x^0\left(\sqrt{x^2-1}\right)^6\right]$

$= 2\left[x^6+15x^4\left(x^2-1\right)+15x^2\left(x^2-1\right)^2+\left(x^2-1\right)^3\right]$

$$= 2\left[x^6 + 15x^6 - 15x^4 + 15x^2\left(x^4 - 2x^2 + 1\right) + \left(x^6 - 1 + 3x^2 - 3x^4\right)\right]$$

$$= 2\left[x^6 + 15x^6 - 15x^4 + 15x^6 - 30x^4 + 15x^2 + x^6 - 1 + 3x^2 - 3x^4\right]$$

$$= 64x^6 - 96x^4 + 36x^2 - 2$$

(iii) $\left(1 + 2\sqrt{x}\right)^5 + \left(1 - 2\sqrt{x}\right)^5$

Let us solve the given expression:
$= 2\ [{}^5C_0\ (2\sqrt{x})^0 + {}^5C_2\ (2\sqrt{x})^2 + {}^5C_4\ (2\sqrt{x})^4]$
$= 2\ [1 + 10\ (4x) + 5\ (16x^2)]$
$= 2\ [1 + 40x + 80x^2]$

(iv) $\left(\sqrt{2} + 1\right)^6 + \left(\sqrt{2} - 1\right)^6$

Let us solve the given expression:
$= 2\ [{}^6C_0\ (\sqrt{2})^6 + {}^6C_2\ (\sqrt{2})^4 + {}^6C_4\ (\sqrt{2})^2 + {}^6C_6\ (\sqrt{2})^0]$
$= 2\ [8 + 15\ (4) + 15\ (2) + 1]$
$= 2\ [99]$
$= 198$

(v) $\left(3 + \sqrt{2}\right)^5 - \left(3 - \sqrt{2}\right)^5$

Let us solve the given expression:
$= 2\ [{}^5C_1\ (3^4)\ (\sqrt{2})^1 + {}^5C_3\ (3^2)\ (\sqrt{2})^3 + {}^5C_5\ (3^0)\ (\sqrt{2})^5]$
$= 2\ [5\ (81)\ (\sqrt{2}) + 10\ (9)\ (2\sqrt{2}) + 4\sqrt{2}]$
$= 2\sqrt{2}\ (405 + 180 + 4)$
$= 1178\sqrt{2}$

(vi) $\left(2 + \sqrt{3}\right)^7 + \left(2 - \sqrt{3}\right)^7$

Let us solve the given expression:
$= 2\ [{}^7C_0\ (2^7)\ (\sqrt{3})^0 + {}^7C_2\ (2^5)\ (\sqrt{3})^2 + {}^7C_4\ (2^3)\ (\sqrt{3})^4 + {}^7C_6\ (2^1)\ (\sqrt{3})^6]$
$= 2\ [128 + 21\ (32)(3) + 35(8)(9) + 7(2)(27)]$
$= 2\ [128 + 2016 + 2520 + 378]$
$= 2\ [5042]$
$= 10084$

(vii) $\left(\sqrt{3}+1\right)^5-\left(\sqrt{3}-1\right)^5$

Let us solve the given expression:

$= 2\ [^5C_1\ (\sqrt{3})^4 + {}^5C_3\ (\sqrt{3})^2 + {}^5C_5\ (\sqrt{3})^0]$

$= 2\ [5\ (9) + 10\ (3) + 1]$

$= 2\ [76]$

$= 152$

(viii) $(0.99)^5+(1.01)^5$

Let us solve the given expression:

$= (1 – 0.01)^5 + (1 + 0.01)^5$

$= 2\ [^5C_0\ (0.01)^0 + {}^5C_2\ (0.01)^2 + {}^5C_4\ (0.01)^4]$

$= 2\ [1 + 10\ (0.0001) + 5\ (0.00000001)]$

$= 2\ [1.00100005]$

$= 2.0020001$

(ix) $\left(\sqrt{3}+\sqrt{2}\right)^6-\left(\sqrt{3}-\sqrt{2}\right)^6$

Let us solve the given expression:

$= 2\ [^6C_1\ (\sqrt{3})^5\ (\sqrt{2})^1 + {}^6C_3\ (\sqrt{3})^3\ (\sqrt{2})^3 + {}^6C_5\ (\sqrt{3})^1\ (\sqrt{2})^5]$

$= 2\ [6\ (9\sqrt{3})\ (\sqrt{2}) + 20\ (3\sqrt{3})\ (2\sqrt{2}) + 6\ (\sqrt{3})\ (4\sqrt{2})]$

$= 2\ [\sqrt{6}\ (54 + 120 + 24)]$

$= 396\ \sqrt{6}$

(x) $\left\{a^2+\sqrt{a^2-1}\right\}^4+\left\{a^2-\sqrt{a^2-1}\right\}^4$

Let us solve the given expression:

$$= 2\left[{}^4C_0\left(a^2\right)^4\left(\sqrt{a^2-1}\right)^0+{}^4C_2\left(a^2\right)^2\left(\sqrt{a^2-1}\right)^2+{}^4C_4\left(a^2\right)^0\left(\sqrt{a^2-1}\right)^4\right]$$

$$= 2\left[a^8+6a^4\left(a^2-1\right)+\left(a^2-1\right)^2\right]$$

$= 2\ [a^8 + 6a^6 – 6a^4 + a^4 + 1 – 2a^2]$

$= 2a^8 + 12a^6 – 10a^4 – 4a^2 + 2$

3. Find $(a + b)^4 – (a – b)^4$. Hence, evaluate $(\sqrt{3} + \sqrt{2})^4 – (\sqrt{3} - \sqrt{2})^4$.

Solution:

Firstly, let us solve the given expression:

$(a + b)^4 – (a – b)^4$

The above expression can be expressed as,
$(a + b)^4 - (a - b)^4 = 2 [{}^4C_1 a^3b^1 + {}^4C_3 a^1b^3]$
$= 2 [4a^3b + 4ab^3]$
$= 8 (a^3b + ab^3)$
Now,
Let us evaluate the expression:
$(\sqrt{3} + \sqrt{2})^4 - (\sqrt{3} - \sqrt{2})^4$
So consider, $a = \sqrt{3}$ and $b = \sqrt{2}$ we get,
$(\sqrt{3} + \sqrt{2})^4 - (\sqrt{3} - \sqrt{2})^4 = 8 (a^3b + ab^3)$
$= 8 [(\sqrt{3})^3 (\sqrt{2}) + (\sqrt{3}) (\sqrt{2})^3]$
$= 8 [(3\sqrt{6}) + (2\sqrt{6})]$
$= 8 (5\sqrt{6})$
$= 40\sqrt{6}$

4. Find $(x + 1)^6 + (x - 1)^6$. Hence, or otherwise evaluate $(\sqrt{2} + 1)^6 + (\sqrt{2} - 1)^6$.
Solution:
Firstly, let us solve the given expression:
$(x + 1)^6 + (x - 1)^6$
The above expression can be expressed as,
$(x + 1)^6 + (x - 1)^6 = 2 [{}^6C_0 x^6 + {}^6C_2 x^4 + {}^6C_4 x^2 + {}^6C_6 x^0]$
$= 2 [x^6 + 15x^4 + 15x^2 + 1]$
Now,
Let us evaluate the expression:
$(\sqrt{2} + 1)^6 + (\sqrt{2} - 1)^6$
So consider, $x = \sqrt{2}$ then we get,
$(\sqrt{2} + 1)^6 + (\sqrt{2} - 1)^6 = 2 [x^6 + 15x^4 + 15x^2 + 1]$
$= 2 [(\sqrt{2})^6 + 15 (\sqrt{2})^4 + 15 (\sqrt{2})^2 + 1]$
$= 2 [8 + 15 (4) + 15 (2) + 1]$
$= 2 [8 + 60 + 30 + 1]$
$= 198$

5. Using binomial theorem evaluate each of the following:
(i) $(96)^3$
(ii) $(102)^5$
(iii) $(101)^4$
(iv) $(98)^5$
Solution:
(i) $(96)^3$
We have,

$(96)^3$
Let us express the given expression as two different entities and apply the binomial theorem.
$(96)^3 = (100 - 4)^3$
$= {}^3C_0 (100)^3 (4)^0 - {}^3C_1 (100)^2 (4)^1 + {}^3C_2 (100)^1 (4)^2 - {}^3C_3 (100)^0 (4)^3$
$= 1000000 – 120000 + 4800 – 64$
$= 884736$

(ii) $(102)^5$
We have,
$(102)^5$
Let us express the given expression as two different entities and apply the binomial theorem.
$(102)^5 = (100 + 2)^5$
$= {}^5C_0 (100)^5 (2)^0 + {}^5C_1 (100)^4 (2)^1 + {}^5C_2 (100)^3 (2)^2 + {}^5C_3 (100)^2 (2)^3 + {}^5C_4 (100)^1 (2)^4 + {}^5C_5 (100)^0 (2)^5$
$= 10000000000 + 1000000000 + 40000000 + 800000 + 8000 + 32$
$= 11040808032$

(iii) $(101)^4$
We have,
$(101)^4$
Let us express the given expression as two different entities and apply the binomial theorem.
$(101)^4 = (100 + 1)^4$
$= {}^4C_0 (100)^4 + {}^4C_1 (100)^3 + {}^4C_2 (100)^2 + {}^4C_3 (100)^1 + {}^4C_4 (100)^0$
$= 100000000 + 4000000 + 60000 + 400 + 1$
$= 104060401$

(iv) $(98)^5$
We have,
$(98)^5$
Let us express the given expression as two different entities and apply the binomial theorem.
$(98)^5 = (100 - 2)^5$
$= {}^5C_0 (100)^5 (2)^0 - {}^5C_1 (100)^4 (2)^1 + {}^5C_2 (100)^3 (2)^2 - {}^5C_3 (100)^2 (2)^3 + {}^5C_4 (100)^1 (2)^4 - {}^5C_5 (100)^0 (2)^5$
$= 10000000000 - 1000000000 + 40000000 - 800000 + 8000 – 32$
$= 9039207968$

6. Using binomial theorem, prove that $2^{3n} - 7n - 1$ is divisible by 49, where $n \in N$.

Solution:

Given:

$2^{3n} - 7n - 1$

So, $2^{3n} - 7n - 1 = 8^n - 7n - 1$

Now,

$8^n - 7n - 1$

$8^n = 7n + 1$

$= (1 + 7)^n$

$= {}^nC_0 + {}^nC_1 (7)^1 + {}^nC_2 (7)^2 + {}^nC_3 (7)^3 + {}^nC_4 (7)^2 + {}^nC_5 (7)^1 + \dots + {}^nC_n (7)^n$

$8^n = 1 + 7n + 49 [{}^nC_2 + {}^nC_3 (7^1) + {}^nC_4 (7^2) + \dots + {}^nC_n (7)^{n-2}]$

$8^n - 1 - 7n = 49$ (integer)

So now,

$8^n - 1 - 7n$ is divisible by 49

Or

$2^{3n} - 1 - 7n$ is divisible by 49.

Hence proved.

EXERCISE 18.2 PAGE NO: 18.37

1. Find the 11^{th} term from the beginning and the 11^{th} term from the end in the expansion of $(2x – 1/x^2)^{25}$.

Solution:

Given:

$(2x – 1/x^2)^{25}$

The given expression contains 26 terms.

So, the 11^{th} term from the end is the $(26 − 11 + 1)^{th}$ term from the beginning.

In other words, the 11^{th} term from the end is the 16^{th} term from the beginning.

Then,

$T_{16} = T_{15+1} = {}^{25}C_{15}\ (2x)^{25-15}\ (-1/x^2)^{15}$

$= {}^{25}C_{15}\ (2^{10})\ (x)^{10}\ (-1/x^{30})$

$= -\ {}^{25}C_{15}\ (2^{10} / x^{20})$

Now we shall find the 11^{th} term from the beginning.

$T_{11} = T_{10+1} = {}^{25}C_{10}\ (2x)^{25-10}\ (-1/x^2)^{10}$

$= {}^{25}C_{10}\ (2^{15})\ (x)^{15}\ (1/x^{20})$

$= {}^{25}C_{10}\ (2^{15} / x^5)$

2. Find the 7^{th} term in the expansion of $(3x^2 – 1/x^3)^{10}$.

Solution:

Given:

$(3x^2 – 1/x^3)^{10}$

Let us consider the 7^{th} term as T_7

So,

$T_7 = T_{6+1}$

$= {}^{10}C_6\ (3x^2)^{10-6}\ (-1/x^3)^6$

$= {}^{10}C_6\ (3)^4\ (x)^8\ (1/x^{18})$

$= [10×9×8×7×81] / [4×3×2×x^{10}]$

$= 17010 / x^{10}$

∴ The 7^{th} term of the expression $(3x^2 – 1/x^3)^{10}$ is $17010 / x^{10}$.

3. Find the 5^{th} term in the expansion of $(3x – 1/x^2)^{10}$.

Solution:

Given:

$(3x – 1/x^2)^{10}$

The 5^{th} term from the end is the $(11 – 5 + 1)$th, is., 7^{th} term from the beginning.

So,

$T_7 = T_{6+1}$
$= {}^{10}C_6\ (3x)^{10-6}\ (-1/x^2)^6$
$= {}^{10}C_6\ (3)^4\ (x)^4\ (1/x^{12})$
$= [10\times9\times8\times7\times81] / [4\times3\times2\times x^8]$
$= 17010 / x^8$
∴ The 5^{th} term of the expression $(3x – 1/x^2)^{10}$ is $17010 / x^8$.

4. Find the 8^{th} term in the expansion of $(x^{3/2}\ y^{1/2} – x^{1/2}\ y^{3/2})^{10}$.
Solution:
Given:
$(x^{3/2}\ y^{1/2} – x^{1/2}\ y^{3/2})^{10}$
Let us consider the 8^{th} term as T_8
So,
$T_8 = T_{7+1}$
$= {}^{10}C_7\ (x^{3/2}\ y^{1/2})^{10-7}\ (-x^{1/2}\ y^{3/2})^7$
$= -[10\times9\times8]/[3\times2]\ x^{9/2}\ y^{3/2}\ (x^{7/2}\ y^{21/2})$
$= -120\ x^8y^{12}$
∴ The 8^{th} term of the expression $(x^{3/2}\ y^{1/2} – x^{1/2}\ y^{3/2})^{10}$ is $-120\ x^8y^{12}$.

5. Find the 7^{th} term in the expansion of $(4x/5 + 5/2x)^8$.
Solution:
Given:
$(4x/5 + 5/2x)^8$
Let us consider the 7^{th} term as T_7
So,
$T_7 = T_{6+1}$
$$= {}^8C_6\left(\frac{4x}{5}\right)^{8-6}\left(\frac{5}{2x}\right)^6$$
$$= \frac{8\times7\times4\times4\times125\times125}{2\times1\times25\times64}\ x^2\ \left(\frac{1}{x^6}\right)$$
$$= \frac{4375}{x^4}$$
∴ The 7^{th} term of the expression $(4x/5 + 5/2x)^8$ is $4375/x^4$.

6. Find the 4^{th} term from the beginning and 4^{th} term from the end in the expansion of $(x + 2/x)^9$.
Solution:
Given:
$(x + 2/x)^9$
Let T_{r+1} be the 4th term from the end.

Then, T_{r+1} is (10 – 4 + 1)th, i.e., 7th, term from the beginning.

$T_7 = T_{6-1}$

$$= {}^9C_6\ (x^{9-6})\left(\frac{2}{x}\right)^6$$

$$= \frac{9\times8\times7}{3\times2}\left(x^3\right)\left(\frac{64}{x^6}\right)$$

$$= \frac{5376}{x^3}$$

4th term from the beginning = $T_4 = T_{3+1}$

$$T_4 = {}^9C_3\ (\ x^{9-3})\ \left(\frac{2}{x}\right)^3$$

$$= \frac{9\times8\times7}{3\times2}\left(x^6\right)\left(\frac{8}{x^3}\right)$$

$= 672\ x^3$

7. Find the 4th term from the end in the expansion of $(4x/5 – 5/2x)^8$.

Solution:

Given:

$(4x/5 – 5/2x)^8$

Let T_{r+1} be the 4th term from the end of the given expression.

Then, T_{r+1} is (10 – 4 + 1)th term, i.e., 7th term, from the beginning.

$T_7 = T_{6+1}$

$$= {}^9C_6\ \left(\frac{4x}{5}\right)^{9-6}\left(\frac{5}{2x}\right)^6$$

$$= \frac{9\times8\times7}{3\times2}\left(\frac{64}{125}x^3\right)\left(\frac{125\times125}{64x^6}\right)$$

$$= \frac{10500}{x^3}$$

∴ The 4th term from the end is $10500/x^3$.

8. Find the 7th term from the end in the expansion of $(2x^2 – 3/2x)^8$.

Solution:

Given:

$(2x^2 – 3/2x)^8$

Let T_{r+1} be the 4th term from the end of the given expression.

Then, T_{r+1} is (9 – 7 + 1)th term, i.e., 3rd term, from the beginning.

$T_3 = T_{2+1}$

$$= {}^8C_2\ \left(2x^2\right)^{8-2}\ \left(-\frac{3}{2x}\right)^2$$

$$= \frac{8\times7}{2\times1}\left(64x^{12}\right)\frac{9}{4x^2}$$

$$= 4032\ x^{10}$$

∴ The 7^{th} term from the end is $4032\ x^{10}$.

9. Find the coefficient of:
(i) x^{10} in the expansion of $(2x^2 – 1/x)^{20}$
(ii) x^7 in the expansion of $(x – 1/x^2)^{40}$
(iii) x^{-15} in the expansion of $(3x^2 – a/3x^3)^{10}$
(iv) x^9 in the expansion of $(x^2 – 1/3x)^9$
(v) x^m in the expansion of $(x + 1/x)^n$
(vi) x in the expansion of $(1 – 2x^3 + 3x^5)(1 + 1/x)^8$
(vii) a^5b^7 in the expansion of $(a – 2b)^{12}$
(viii) x in the expansion of $(1 – 3x + 7x^2)(1 - x)^{16}$
Solution:
(i) x^{10} in the expansion of $(2x^2 – 1/x)^{20}$
Given:
$(2x^2 – 1/x)^{20}$
If x^{10} occurs in the (r + 1)th term in the given expression.
Then, we have:
$T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$

$$T_{r+1} = {}^{20}C_r \left(2x^2\right)^{20-r} \left(\frac{-1}{x}\right)^r$$
$$= \left(-1\right)^r\, {}^{20}C_r\, \left(2^{20-r}\right)\left(x^{40-2r-r}\right)$$

For this term to contain x^{10}, we must have:
40 − 3r =10
3r = 30
r = 10

$$\therefore \text{ Coefficient of } x^{10} = \left(-1\right)^{10}\, {}^{20}C_{10}\left(2^{20-10}\right) = {}^{20}C_{10}\left(2^{10}\right)$$

(ii) x^7 in the expansion of $(x – 1/x^2)^{40}$
Given:
$(x – 1/x^2)^{40}$
If x^7 occurs at the (r + 1) th term in the given expression.
Then, we have:
$T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$

$$T_{r+1} = {}^{40}C_r\, x^{40-r}\left(\frac{-1}{x^2}\right)^r$$
$$= \left(-1\right)^r\, {}^{40}C_r\, x^{40-r-2r}$$

For this term to contain x^7, we must have:

40 − 3r =7
3r = 40 – 7
3r = 33
r = 33/3
= 11

$\therefore$ Coefficient of $x^7 = (-1)^{11}\ {}^{40}C_{11} = -{}^{40}C_{11}$

(iii) x^{-15} in the expansion of $(3x^2 – a/3x^3)^{10}$
Given:
$(3x^2 – a/3x^3)^{10}$
If x^{-15} occurs at the (r + 1)th term in the given expression.
Then, we have:
$T_{r+1} = {}^nC_r\ x^{n-r}\ a^r$

$$T_{r+1} = {}^{10}C_r\ (3x^2)^{10-r} \left(\frac{-a}{3x^3}\right)^r$$
$$= (-1)^r\ {}^{10}C_r\ (3^{10-r-r})\ (x^{20-2r-3r})\ (a^r)$$

For this term to contain x^{-15}, we must have:
20 − 5r = -15
5r = 20 + 15
5r = 35
r = 35/5
= 7

$\therefore$ Coefficient of $x^{-15} = (-1)^7\ {}^{10}C_7\ 3^{10-14}\ (a^7) = -\frac{10\times9\times8}{3\times2\times9\times9}a^7 = -\frac{40}{27}a^7$

(iv) x^9 in the expansion of $(x^2 – 1/3x)^9$
Given:
$(x^2 – 1/3x)^9$
If x^9 occurs at the (r + 1)th term in the above expression.
Then, we have:
$T_{r+1} = {}^nC_r\ x^{n-r}\ a^r$

$$T_{r+1} = {}^9C_r\ (x^2)^{9-r} \left(\frac{-1}{3x}\right)^r$$
$$= (-1)^r\ {}^9C_r\ (x^{18-2r-r})\ \left(\frac{1}{3^r}\right)$$

For this term to contain x^9, we must have:
18 – 3r = 9
3r = 18 - 9

3r = 9
r = 9/3
= 3

$\therefore \text{Coefficient of } x^9 = (-1)^3 \, {}^9C_3 \frac{1}{3^3} = -\frac{9\times8\times7}{2\times9\times9} = \frac{-28}{9}$

(v) x^m in the expansion of $(x + 1/x)^n$
Given:
$(x + 1/x)^n$
If x^m occurs at the (r + 1)th term in the given expression.
Then, we have:
$T_{r+1} = {}^nC_r x^{n-r} a^r$

$T_{r+1} = {}^nC_r \, x^{n-r} \frac{1}{x^r}$

$= {}^nC_r \, x^{n-2r}$

For this term to contain x^m, we must have:
n – 2r = m
2r = n – m
r = (n - m)/2

$\therefore \text{Coefficient of } x^m = {}^nC_{(n-m)/2} = \frac{n!}{\left(\frac{n-m}{2}\right)! \left(\frac{n+m}{2}\right)!}$

(vi) x in the expansion of $(1 – 2x^3 + 3x^5) (1 + 1/x)^8$
Given:
$(1 – 2x^3 + 3x^5) (1 + 1/x)^8$
If x occurs at the (r + 1)th term in the given expression.
Then, we have:
$(1 – 2x^3 + 3x^5) (1 + 1/x)^8 = (1 – 2x^3 + 3x^5) ({}^8C_0 + {}^8C_1 (1/x) + {}^8C_2 (1/x)^2 + {}^8C_3 (1/x)^3 + {}^8C_4 (1/x)^4 + {}^8C_5 (1/x)^5 + {}^8C_6 (1/x)^6 + {}^8C_7 (1/x)^7 + {}^8C_8 (1/x)^8)$
So, 'x' occurs in the above expression at $-2x^3.{}^8C_2 (1/x^2) + 3x^5.{}^8C_4 (1/x^4)$
∴ Coefficient of x = -2 (8!/(2!6!)) + 3 (8!/(4! 4!))
= -56 + 210
= 154

(vii) a^5b^7 in the expansion of $(a – 2b)^{12}$
Given:
$(a – 2b)^{12}$
If a^5b^7 occurs at the (r + 1)th term in the given expression.

Then, we have:
$T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$

$$T_{r+1} = {}^{12}C_r\, a^{12-r}\,(-2b)^r$$
$$= (-1)^r\, {}^{12}C_r\,(a^{12-r})\,(b^r)\,(2^r)$$

For this term to contain a^5b^7, we must have:

$12 - r = 5$
$r = 12 - 5$
$= 7$

$$\therefore \text{ Required coefficient } = (-1)^7\, {}^{12}C_7\,(2^7)$$
$$= -\frac{12\times11\times10\times9\times8\times128}{5\times4\times3\times2}$$
$$= -101376$$

(viii) x in the expansion of $(1 - 3x + 7x^2)(1 - x)^{16}$
Given:
$(1 - 3x + 7x^2)(1 - x)^{16}$
If x occurs at the (r + 1)th term in the given expression.
Then, we have:
$(1 - 3x + 7x^2)(1 - x)^{16} = (1 - 3x + 7x^2)({}^{16}C_0 + {}^{16}C_1(-x) + {}^{16}C_2(-x)^2 + {}^{16}C_3(-x)^3 + {}^{16}C_4(-x)^4 + {}^{16}C_5(-x)^5 + {}^{16}C_6(-x)^6 + {}^{16}C_7(-x)^7 + {}^{16}C_8(-x)^8 + {}^{16}C_9(-x)^9 + {}^{16}C_{10}(-x)^{10} + {}^{16}C_{11}(-x)^{11} + {}^{16}C_{12}(-x)^{12} + {}^{16}C_{13}(-x)^{13} + {}^{16}C_{14}(-x)^{14} + {}^{16}C_{15}(-x)^{15} + {}^{16}C_{16}(-x)^{16})$
So, 'x' occurs in the above expression at ${}^{16}C_1(-x) - 3x\,{}^{16}C_0$
$\therefore$ Coefficient of x = -(16!/(1! 15!)) – 3(16!/(0! 16!))
= -16 – 3
= -19

10. Which term in the expansion of $\left\{\left(\frac{x}{\sqrt{y}}\right)^{1/3} + \left(\frac{y}{x^{1/3}}\right)^{1/2}\right\}^{21}$ **contains x and y to one and the same power.**

Solution:

Let us consider T_{r+1} th term in the given expansion contains x and y to one and the same power.
Then we have,
$T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$

$$T_{r+1} = {}^{21}C_r\left[\left(\frac{x}{\sqrt{y}}\right)^{1/3}\right]^{21-r}\left[\left(\frac{y}{x^{1/3}}\right)^{1/2}\right]^r$$
$$= {}^{21}C_r\left(\frac{x^{(21-r)/3}}{x^{r/6}}\right)\left(\frac{y^{r/2}}{y^{(21-r)/6}}\right)$$

$$= {}^{21}C_r\ (x)^{7-r/2} (y)^{2r/3-7/2}$$

If x and y have the same power, then

7 – r/2 = 2r/3 – 7/2

2r/3 + r/2 = 7 + 7/2

(4r + 3r)/6 = (14+7)/2

7r/6 = 21/2

r = (21 × 6)/(2 × 7)

= 3×3

= 9

Hence, the required term is the 10^{th} term.

11. Does the expansion of $(2x^2 – 1/x)$ contain any term involving x^9?

Solution:

Given:

$(2x^2 – 1/x)$

If x^9 occurs at the (r + 1)th term in the given expression.

Then, we have:

$T_{r+1} = {}^nC_r\ x^{n-r}\ a^r$

$$T_{r+1} = {}^{20}C_r\ \left(2x^2\right)^{20-r}\ \left(\frac{-1}{x}\right)^r$$

$$= \left(-1\right)^r\ {}^{20}C_r (2)^{20-r}\ (x)^{40-2r-r}$$

For this term to contain x^9, we must have

40 – 3r = 9

3r = 40 - 9

3r = 31

r = 31/3

It is not possible, since r is not an integer.

Hence, there is no term with x^9 in the given expansion.

12. Show that the expansion of $(x^2 + 1/x)^{12}$ does not contain any term involving x^{-1}.

Solution:

Given:

$(x^2 + 1/x)^{12}$

If x^{-1} occurs at the (r + 1)th term in the given expression.

Then, we have:

$T_{r+1} = {}^nC_r\ x^{n-r}\ a^r$

$T_{r+1} = {}^{12}C_r \left(x^2\right)^{12-r} \left(\frac{1}{x}\right)^r$
$= {}^{12}C_r \, x^{24-2r-r}$
For this term to contain x^{-1}, we must have
24 – 3r = -1
3r = 24 + 1
3r = 25
r = 25/3
It is not possible, since r is not an integer.
Hence, there is no term with x^{-1} in the given expansion.

13. Find the middle term in the expansion of:
(i) $(2/3x - 3/2x)^{20}$
(ii) $(a/x + bx)^{12}$
(iii) $(x^2 - 2/x)^{10}$
(iv) $(x/a - a/x)^{10}$
Solution:
(i) $(2/3x - 3/2x)^{20}$
We have,
$(2/3x - 3/2x)^{20}$ where, n = 20 (even number)
So the middle term is (n/2 + 1) = (20/2 + 1) = (10 + 1) = 11. ie., 11^{th} term
Now,
$T_{11} = T_{10+1}$
$= {}^{20}C_{10} (2/3x)^{20-10} (3/2x)^{10}$
$= {}^{20}C_{10}\, 2^{10}/3^{10} \times 3^{10}/2^{10}\, x^{10-10}$
$= {}^{20}C_{10}$
Hence, the middle term is ${}^{20}C_{10}$.

(ii) $(a/x + bx)^{12}$
We have,
$(a/x + bx)^{12}$ where, n = 12 (even number)
So the middle term is (n/2 + 1) = (12/2 + 1) = (6 + 1) = 7. ie., 7^{th} term
Now,
$T_7 = T_{6+1}$
$= {}^{12}C_6 \left(\frac{a}{x}\right)^{12-6} \left(bx\right)^6$
$= {}^{12}C_6 \, a^6 \, b^6$
$= \frac{12\times11\times10\times9\times8\times7}{6\times5\times4\times3\times2} a^6 \, b^6$

$= 924\ a^6b^6$

Hence, the middle term is $924\ a^6b^6$.

(iii) $(x^2 – 2/x)^{10}$

We have,

$(x^2 – 2/x)^{10}$ where, n = 10 (even number)

So the middle term is (n/2 + 1) = (10/2 + 1) = (5 + 1) = 6. ie., 6^{th} term

Now,

$T_6 = T_{5+1}$

$$= {}^{10}C_5 \left(x^2\right)^{10-5} \left(\frac{-2}{x}\right)^5$$

$$= -\frac{10\times9\times8\times7\times6}{5\times4\times3\times2} \times 32x^5$$

$= -\ 8064\ x^5$

Hence, the middle term is $-8064x^5$.

(iv) $(x/a – a/x)^{10}$

We have,

$(x/a – a/x)^{10}$ where, n = 10 (even number)

So the middle term is (n/2 + 1) = (10/2 + 1) = (5 + 1) = 6. ie., 6^{th} term

Now,

$T_6 = T_{5+1}$

$$= {}^{10}C_5 \left(\frac{x}{a}\right)^{10-5} \left(\frac{-a}{x}\right)^5$$

$$= -\frac{10\times9\times8\times7\times6}{5\times4\times3\times2}$$

$= -\ 252$

Hence, the middle term is -252.

14. Find the middle terms in the expansion of:

(i) $(3x – x^3/6)^9$

(ii) $(2x^2 – 1/x)^7$

(iii) $(3x – 2/x^2)^{15}$

(iv) $(x^4 – 1/x^3)^{11}$

Solution:

(i) $(3x – x^3/6)^9$

We have,

$(3x – x^3/6)^9$ where, n = 9 (odd number)

So the middle terms are ((n+1)/2) = ((9+1)/2) = 10/2 = 5 and ((n+1)/2 + 1) = ((9+1)/2 + 1) = (10/2 + 1) = (5 + 1) = 6

The terms are 5^{th} and 6^{th}.

Now,
$T_5 = T_{4+1}$

$$= {}^9C_4 \left(3x\right)^{9-4} \left(\frac{-x^3}{6}\right)^4$$
$$= \frac{9\times8\times7\times6}{4\times3\times2} \times 27 \times 9 \times \frac{1}{36\times36} x^{17}$$
$$= \frac{189}{8} x^{17}$$

And,
$T_6 = T_{5+1}$

$$= {}^9C_5 \left(3x\right)^{9-5} \left(\frac{-x^3}{6}\right)^5$$
$$= -\frac{9\times8\times7\times6}{4\times3\times2} \times 81 \times \frac{1}{216\times36} x^{19}$$
$$= -\frac{21}{16} x^{19}$$

Hence, the middle term are 189/8 x^{17} and -21/16 x^{19}.

(ii) $(2x^2 – 1/x)^7$
We have,
$(2x^2 – 1/x)^7$ where, n = 7 (odd number)
So the middle terms are ((n+1)/2) = ((7+1)/2) = 8/2 = 4 and ((n+1)/2 + 1) = ((7+1)/2 + 1) = (8/2 + 1) = (4 + 1) = 5
The terms are 4th and 5th.
Now,
$T_4 = T_{3-1}$

$$= {}^7C_3 \left(2x^2\right)^{7-3} \left(\frac{-1}{x}\right)^3$$
$$= -\frac{7\times6\times5}{3\times2} \times 16\, x^8 \times \frac{1}{x^3}$$
$$= -560\, x^5$$

And,
$T_5 = T_{4-1}$

$$= {}^7C_4 \left(2x^2\right)^{7-4} \left(\frac{-1}{x}\right)^4$$
$$= 35 \times 8 \times x^6 \times \frac{1}{x^4}$$
$$= 280\, x^2$$

Hence, the middle term are $-560x^5$ and $280x^2$.

(iii) $(3x - 2/x^2)^{15}$
We have,
$(3x - 2/x^2)^{15}$ where, n = 15 (odd number)
So the middle terms are ((n+1)/2) = ((15+1)/2) = 16/2 = 8 and
((n+1)/2 + 1) = ((15+1)/2 + 1) = (16/2 + 1) = (8 + 1) = 9
The terms are 8th and 9th.
Now,
$T_8 = T_{7-1}$

$$= {}^{15}C_7 \left(3x\right)^{15-7} \left(\frac{-2}{x^2}\right)^7$$

$$= -\frac{15\times14\times13\times12\times11\times10\times9}{7\times6\times5\times4\times3\times2} \times 3^8 \times 2^7\ x^{8-14}$$

$$= \frac{-6435\times3^8\times2^7}{x^6}$$

And,
$T_9 = T_{8-1}$

$$= {}^{15}C_8 \left(3x\right)^{15-8} \left(\frac{-2}{x^2}\right)^8$$

$$= \frac{15\times14\times13\times12\times11\times10\times9}{7\times6\times5\times4\times3\times2} \times 3^7 \times 2^8 \times x^{7-16}$$

$$= \frac{6435\times3^7\times2^8}{x^9}$$

Hence, the middle term are $(-6435\times3^8\times2^7)/x^6$ and $(6435\times3^7\times2^8)/x^9$.

(iv) $(x^4 - 1/x^3)^{11}$
We have,
$(x^4 - 1/x^3)^{11}$
where, n = 11 (odd number)
So the middle terms are ((n+1)/2) = ((11+1)/2) = 12/2 = 6 and
((n+1)/2 + 1) = ((11+1)/2 + 1) = (12/2 + 1) = (6 + 1) = 7
The terms are 6th and 7th.
Now,
$T_6 = T_{5-1}$

$$= {}^{11}C_5 \left(x^4\right)^{11-5} \left(\frac{-1}{x^3}\right)^5$$

$$= -\frac{11\times10\times9\times8\times7}{5\times4\times3\times2} \times (x)^{24-15}$$

$$= -462\ x^9$$

And,
$T_7 = T_{6+1}$

$= {}^{11}C_6 \left(x^4\right)^{11-6} \left(\frac{-1}{x^3}\right)^6$

$= \frac{11\times10\times9\times8\times7}{5\times4\times3\times2}(x)^{20-18}$

$= 462\ x^2$

Hence, the middle term are $-462x^9$ and $462x^2$.

15. Find the middle terms in the expansion of:
(i) $(x - 1/x)^{10}$
(ii) $(1 - 2x + x^2)^n$
(iii) $(1 + 3x + 3x^2 + x^3)^{2n}$
(iv) $(2x - x^2/4)^9$
(v) $(x - 1/x)^{2n+1}$
(vi) $(x/3 + 9y)^{10}$
(vii) $(3 - x^3/6)^7$
(viii) $(2ax - b/x^2)^{12}$
(ix) $(p/x + x/p)^9$
(x) $(x/a - a/x)^{10}$
Solution:
(i) $(x - 1/x)^{10}$
We have,
$(x - 1/x)^{10}$ where, n = 10 (even number)
So the middle term is (n/2 + 1) = (10/2 + 1) = (5 + 1) = 6. ie., 6th term
Now,
$T_6 = T_{5+1}$

$= {}^{10}C_5\ x^{10-5} \left(\frac{-1}{x}\right)^5$

$= -\frac{10\times9\times8\times7\times6}{5\times4\times3\times2}$

$= -252$

Hence, the middle term is -252.

(ii) $(1 - 2x + x^2)^n$
We have,
$(1 - 2x + x^2)^n = (1 - x)^{2n}$ where, n is an even number.
So the middle term is (2n/2 + 1) = (n + 1)th term.
Now,
$T_n = T_{n+1}$
$= {}^{2n}C_n (-1)^n (x)^n$
$= (2n)!/(n!)^2 (-1)^n x^n$

Hence, the middle term is $(2n)!/(n!)^2 (-1)^n x^n$.

(iii) $(1 + 3x + 3x^2 + x^3)^{2n}$
We have,
$(1 + 3x + 3x^2 + x^3)^{2n} = (1 + x)^{6n}$ where, n is an even number.
So the middle term is (n/2 + 1) = (6n/2 + 1) = (3n + 1)th term.
Now,
$T_{2n} = T_{3n+1}$
$= {}^{6n}C_{3n}\, x^{3n}$
$= (6n)!/(3n!)^2\, x^{3n}$
Hence, the middle term is $(6n)!/(3n!)^2 x^{3n}$.

(iv) $(2x - x^2/4)^9$
We have,
$(2x - x^2/4)^9$ where, n = 9 (odd number)
So the middle terms are ((n+1)/2) = ((9+1)/2) = 10/2 = 5 and
((n+1)/2 + 1) = ((9+1)/2 + 1) = (10/2 + 1) = (5 + 1) = 6
The terms are 5th and 6th.
Now,
$T_5 = T_{4+1}$

$$= {}^9C_4 \left(2x\right)^{9-4} \left(\frac{-x^2}{4}\right)^4$$

$$= \frac{9\times8\times7\times6}{4\times3\times2} \times 2^5\ \frac{1}{4^4} x^{5+8}$$

$$= \frac{63}{4} x^{13}$$

And,
$T_6 = T_{5+1}$

$$= {}^9C_5 \left(2x\right)^{9-5} \left(\frac{-x^2}{4}\right)^5$$

$$= -\frac{9\times8\times7\times6}{4\times3\times2} \times 2^4\ \frac{1}{4^5} x^{4+10}$$

$$= -\frac{63}{32} x^{14}$$

Hence, the middle term is $63/4\ x^{13}$ and $-63/32\ x^{14}$.

(v) $(x - 1/x)^{2n+1}$
We have,
$(x - 1/x)^{2n+1}$ where, n = (2n + 1) is an (odd number)
So the middle terms are ((n+1)/2) = ((2n+1+1)/2) = (2n+2)/2 = (n + 1) and

((n+1)/2 + 1) = ((2n+1+1)/2 + 1) = ((2n+2)/2 + 1) = (n + 1 + 1) = (n + 2)
The terms are $(n + 1)^{th}$ and $(n + 2)^{th}$.
Now,
$T_n = T_{n+1}$

$$= {}^{2n+1}C_n\, x^{2n+1-n} \times \frac{(-1)^n}{x^n}$$

$$= (-1)^n\, {}^{2n+1}C_n\, x$$

And,
$T_{n+2} = T_{n+1+1}$

$$= {}^{2n+1}C_n\, x^{2n+1-n-1}\, \frac{(-1)^{n+1}}{x^{n+1}}$$

$$= (-1)^{n+1}\, {}^{2n+1}C_n \times \frac{1}{x}$$

Hence, the middle term is $(-1)^n . {}^{2n+1}C_n$ x and $(-1)^{n+1} . {}^{2n+1}C_n$ (1/x).

(vi) $(x/3 + 9y)^{10}$
We have,
$(x/3 + 9y)^{10}$ where, n = 10 is an even number.
So the middle term is (n/2 + 1) = (10/2 + 1) = (5 + 1) = 6. i.e., 6th term.
Now,
$T_6 = T_{5+1}$

$$= {}^{10}C_5 \left(\frac{x}{3}\right)^{10-5} (9y)^5$$

$$= \frac{10\times9\times8\times7\times6}{5\times4\times3\times2} \times \frac{1}{3^5} \times 9^5 \times x^5\, y^5$$

$$= \mathbf{61236}\, x^5\, y^5$$

Hence, the middle term is $61236x^5y^5$.

(vii) $(3 - x^3/6)^7$
We have,
$(3 - x^3/6)^7$ where, n = 7 (odd number).
So the middle terms are ((n+1)/2) = ((7+1)/2) = 8/2 = 4 and
((n+1)/2 + 1) = ((7+1)/2 + 1) = (8/2 + 1) = (4 + 1) = 5
The terms are 4^{th} and 5^{th}.
Now,
$T_4 = T_{3+1}$
$= {}^7C_3\, (3)^{7-3}\, (-x^3/6)^3$
$= -105/8\ x^9$
And,
$T_5 = T_{4+1}$

$= {}^9C_4\,(3)^{9-4}\,(-x^3/6)^4$

$= \frac{7\times6\times5}{3\times2} \times 3^5 \times \frac{1}{6^4}\, x^{12}$

$= \frac{35}{48} x^{12}$

Hence, the middle terms are $-105/8\ x^9$ and $35/48\ x^{12}$.

(viii) $(2ax - b/x^2)^{12}$

We have,

$(2ax - b/x^2)^{12}$ where, n = 12 is an even number.

So the middle term is (n/2 + 1) = (12/2 + 1) = (6 + 1) = 7. i.e., 7th term.

Now,

$T_7 = T_{6+1}$

$= {}^{12}C_6\,(2ax)^{12-6}\left(\frac{-b}{x^2}\right)^6$

$= \frac{12\times11\times10\times9\times8\times7}{6\times5\times4\times3\times2\times1} \times \left(\frac{2ab}{x}\right)^6$

$= \frac{59136\,a^6b^6}{x^6}$

Hence, the middle term is $(59136a^6b^6)/x^6$.

(ix) $(p/x + x/p)^9$

We have,

$(p/x + x/p)^9$ where, n = 9 (odd number).

So the middle terms are ((n+1)/2) = ((9+1)/2) = 10/2 = 5 and

((n+1)/2 + 1) = ((9+1)/2 + 1) = (10/2 + 1) = (5 + 1) = 6

The terms are 5^{th} and 6^{th}.

Now,

$T_5 = T_{4+1}$

$= {}^9C_4\left(\frac{p}{x}\right)^{9-4}\left(\frac{x}{p}\right)^4$

$= \frac{9\times8\times7\times6}{4\times3\times2\times1} \times \left(\frac{p}{x}\right)$

$= \frac{126\,p}{x}$

And,

$T_6 = T_{5+1}$

$= {}^9C_5\,(p/x)^{9-5}\,(x/p)^5$

$= \frac{9\times8\times7\times6}{4\times3\times2\times1} \times \left(\frac{x}{p}\right)$

$= \frac{126\,x}{p}$

Hence, the middle terms are 126p/x and 126x/p.

(x) $(x/a - a/x)^{10}$
We have,
$(x/a - a/x)^{10}$ where, n = 10 (even number)
So the middle term is (n/2 + 1) = (10/2 + 1) = (5 + 1) = 6. ie., 6^{th} term
Now,
$T_6 = T_{5+1}$
$= {}^{10}C_5 \left(\frac{x}{a}\right)^{10-5} \left(\frac{-a}{x}\right)^5$
$= -\frac{10\times9\times8\times7\times6}{5\times4\times3\times2}$
= - 252
Hence, the middle term is -252.

16. Find the term independent of x in the expansion of the following expressions:
(i) $(3/2\ x^2 - 1/3x)^9$
(ii) $(2x + 1/3x^2)^9$
(iii) $(2x^2 - 3/x^3)^{25}$
(iv) $(3x - 2/x^2)^{15}$
(v) $((\sqrt{x}/3) + \sqrt{3}/2x^2)^{10}$
(vi) $(x - 1/x^2)^{3n}$
(vii) $(1/2\ x^{1/3} + x^{-1/5})^8$
(viii) $(1 + x + 2x^3)(3/2x^2 - 3/3x)^9$
(ix) $(\sqrt[3]{x} + 1/2\sqrt[3]{x})^{18}$, x > 0
(x) $(3/2x^2 - 1/3x)^6$
Solution:
(i) $(3/2\ x^2 - 1/3x)^9$
Given:
$(3/2\ x^2 - 1/3x)^9$
If (r + 1)th term in the given expression is independent of x.
Then, we have:
$T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$
$= {}^9C_r \left(\frac{3}{2}x^2\right)^{9-r} \left(\frac{-1}{3x}\right)^r$
$= \left(-1\right)^r {}^9C_r \cdot \frac{3^{9-2r}}{2^{9-r}} \times x^{18-2r-r}$
For this term to be independent of x, we must have
18 – 3r = 0
3r = 18
r = 18/3
= 6

So, the required term is 7th term.
We have,
$T_7 = T_{6+1}$
$= {}^9C_6 \times (3^{9-12})/(2^{9-6})$
$= (9\times8\times7)/(3\times2) \times 3^{-3} \times 2^{-3}$
$= 7/18$
Hence, the term independent of x is 7/18.

(ii) $(2x + 1/3x^2)^9$
Given:
$(2x + 1/3x^2)^9$
If (r + 1)th term in the given expression is independent of x.
Then, we have:
$T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$

$$= {}^9C_r \left(2x\right)^{9-r} \left(\frac{1}{3x^2}\right)^r$$

$$= {}^9C_r \cdot \frac{2^{9-r}}{3^r}\, x^{9-r-2r}$$

For this term to be independent of x, we must have
$9 - 3r = 0$
$3r = 9$
$r = 9/3$
$= 3$
So, the required term is 4th term.
We have,
$T_4 = T_{3+1}$
$= {}^9C_3 \times (2^6)/(3^3)$
$= {}^9C_3 \times 64/27$
Hence, the term independent of x is ${}^9C_3 \times 64/27$.

(iii) $(2x^2 - 3/x^3)^{25}$
Given:
$(2x^2 - 3/x^3)^{25}$
If (r + 1)th term in the given expression is independent of x.
Then, we have:
$T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$
$= {}^{25}C_r\, (2x^2)^{25-r}\, (-3/x^3)^r$
$= (-1)^r\, {}^{25}C_r \times 2^{25-r} \times 3^r\, x^{50-2r-3r}$
For this term to be independent of x, we must have

$50 - 5r = 0$
$5r = 50$
$r = 50/5$
$= 10$
So, the required term is 11^{th} term.
We have,
$T_{11} = T_{10+1}$
$= (-1)^{10}\ {}^{25}C_{10} \times 2^{25-10} \times 3^{10}$
$= {}^{25}C_{10}\ (2^{15} \times 3^{10})$
Hence, the term independent of x is ${}^{25}C_{10}\ (2^{15} \times 3^{10})$.

(iv) $(3x - 2/x^2)^{15}$
Given:
$(3x - 2/x^2)^{15}$
If (r + 1)th term in the given expression is independent of x.
Then, we have:
$T_{r+1} = {}^{n}C_r\ x^{n-r}\ a^r$
$= {}^{15}C_r\ (3x)^{15-r}\ (-2/x^2)^r$
$= (-1)^r\ {}^{15}C_r \times 3^{15-r} \times 2^r\ x^{15-r-2r}$
For this term to be independent of x, we must have
$15 - 3r = 0$
$3r = 15$
$r = 15/3$
$= 5$
So, the required term is 6^{th} term.
We have,
$T_6 = T_{5+1}$
$= (-1)^5\ {}^{15}C_5 \times 3^{15-5} \times 2^5$
$= -3003 \times 3^{10} \times 2^5$
Hence, the term independent of x is $-3003 \times 3^{10} \times 2^5$.

(v) $((\sqrt{x}/3) + \sqrt{3}/2x^2)^{10}$
Given:
$((\sqrt{x}/3) + \sqrt{3}/2x^2)^{10}$
If (r + 1)th term in the given expression is independent of x.
Then, we have:
$T_{r+1} = {}^{n}C_r\ x^{n-r}\ a^r$

$$= {}^{10}C_r \left(\sqrt{\frac{x}{3}}\right)^{10-r} \left(\frac{3}{2x^2}\right)^r$$

$$= {}^{10}C_r \cdot \frac{3^{r - \frac{10-r}{2}}}{2^r} x^{\frac{10-r}{2} - 2r}$$

For this term to be independent of x, we must have

$(10-r)/2 - 2r = 0$

$10 - 5r = 0$

$5r = 10$

$r = 10/5$

$= 2$

So, the required term is 3rd term.

We have,

$T_3 = T_{2+1}$

$$= {}^{10}C_2 \times \frac{3^{2 - \frac{10-2}{2}}}{2^2}$$

$$= \frac{10 \times 9}{2 \times 4 \times 9}$$

$= 90/72$

$= 15/12$

$= 5/4$

Hence, the term independent of x is 5/4.

(vi) $(x - 1/x^2)^{3n}$

Given:

$(x - 1/x^2)^{3n}$

If (r + 1)th term in the given expression is independent of x.

Then, we have:

$T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$

$= {}^{3n}C_r\, x^{3n-r}\, (-1/x^2)^r$

$= (-1)^r\, {}^{3n}C_r\, x^{3n-r-2r}$

For this term to be independent of x, we must have

$3n - 3r = 0$

$r = n$

So, the required term is (n+1)th term.

We have,

$(-1)^n\, {}^{3n}C_n$

Hence, the term independent of x $(-1)^n\, {}^{3n}C_n$

(vii) $(1/2\, x^{1/3} + x^{-1/5})^8$

Given:

$(1/2\ x^{1/3} + x^{-1/5})^8$

If (r + 1)th term in the given expression is independent of x.

Then, we have:

$T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$

$$= {}^8C_r \left(\frac{1}{2}x^{1/3}\right)^{8-r} \left(x^{-1/5}\right)^r$$

$$= {}^8C_r.\ \frac{1}{2^{8-r}}\ x^{\frac{8-r}{3}-\frac{r}{5}}$$

For this term to be independent of x, we must have

$(8-r)/3 – r/5 = 0$

$(40 – 5r – 3r)/15 = 0$

$40 – 5r – 3r = 0$

$40 – 8r = 0$

$8r = 40$

$r = 40/8$

$= 5$

So, the required term is 6th term.

We have,

$T_5 = T_{5+1}$

$= {}^8C_5 \times 1/(2^{8-5})$

$= (8\times7\times6)/(3\times2\times8)$

$= 7$

Hence, the term independent of x is 7.

(viii) $(1 + x + 2x^3)\ (3/2x^2 – 3/3x)^9$

Given:

$(1 + x + 2x^3)\ (3/2x^2 – 3/3x)^9$

If (r + 1)th term in the given expression is independent of x.

Then, we have:

$(1 + x + 2x^3)\ (3/2x^2 – 3/3x)^9 =$

$$= (1+x+2x^3)\left[\left(\frac{3}{2}x^2\right)^9 - {}^9C_1\left(\frac{3}{2}x^2\right)^8\frac{1}{3x}\cdots + {}^9C_6\left(\frac{3}{2}x^2\right)^3\left(\frac{1}{3x}\right)^6 - {}^9C_7\left(\frac{3}{2}x^2\right)^2\left(\frac{1}{3x}\right)^7\right.$$

By computing we get,

The term independent of x

$$= 1\left[{}^9C_6\frac{3^3}{2^3}\times\frac{1}{3^6}\right] - 2x^3\left[{}^9C_7\frac{3^3}{2^3}\times\frac{1}{3^7}\times\frac{1}{x^3}\right]$$

$$= \left[\frac{9\times8\times7}{1\times2\times3}\times\frac{1}{8\times27}\right] - 2\left[\frac{9\times8}{1\times2} - \frac{1}{4\times243}\right]$$

= 7/18 – 2/27
= (189 - 36)/486
= 153/486 (divide by 9)
= 17/54
Hence, the term independent of x is 17/54.

(ix) $(\sqrt[3]{x} + 1/2\sqrt[3]{x})^{18}$, x > 0
Given:
$(\sqrt[3]{x} + 1/2\sqrt[3]{x})^{18}$, x > 0
If (r + 1)th term in the given expression is independent of x.
Then, we have:
$T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$

$$= {}^{18}C_r \left(x^{1/3}\right)^{18-r} \left(\frac{1}{2\,x^{1/3}}\right)^r$$

$$= {}^{18}C_r \times \frac{1}{2^r}\, x^{\frac{18-r}{3} - \frac{r}{3}}$$

For this term to be independent of r, we must have
(18-r)/3 – r/3 = 0
(18 – r – r)/3 = 0
18 – 2r = 0
2r = 18
r = 18/2
= 9
So, the required term is 10th term.
We have,
$T_{10} = T_{9+1}$
$= {}^{18}C_9 \times 1/2^9$
Hence, the term independent of x is ${}^{18}C_9 \times 1/2^9$.

(x) $(3/2x^2 – 1/3x)^6$
Given:
$(3/2x^2 – 1/3x)^6$
If (r + 1)th term in the given expression is independent of x.
Then, we have:
$T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$

$$= {}^6C_r \left(\frac{3}{2}x^2\right)^{6-r} \left(\frac{-1}{3x}\right)^r$$

$$= (-1)^r\, {}^6C_r \times \frac{3^{6-r-r}}{2^{6-r}}\, x^{12-2r-r}$$

For this term to be independent of r, we must have

12 – 3r = 0
3r = 12
r = 12/3
= 4
So, the required term is 5th term.
We have,
$T_5 = T_{4+1}$

$$= {}^6C_4 \times \frac{3^{6-4-4}}{2^{6-4}}$$

$$= \frac{6\times5}{2\times1\times4\times9}$$

$$= \frac{5}{12}$$

Hence, the term independent of x is 5/12.

17. If the coefficients of (2r + 4)th and (r - 2)th terms in the expansion of $(1 + x)^{18}$ are equal, find r.

Solution:

Given:
$(1 + x)^{18}$
We know, the coefficient of the r term in the expansion of $(1 + x)^n$ is ${}^nC_{r-1}$
So, the coefficients of the (2r + 4) and (r - 2) terms in the given expansion are ${}^{18}C_{2r+4-1}$ and ${}^{18}C_{r-2-1}$
For these coefficients to be equal, we must have
${}^{18}C_{2r+4-1} = {}^{18}C_{r-2-1}$
${}^{18}C_{2r+3} = {}^{18}C_{r-3}$
2r + 3 = r – 3 (or) 2r + 3 + r – 3 = 18 [Since, ${}^nC_r = {}^nC_s$ => r = s (or) r + s = n]
2r – r = -3 – 3 (or) 3r = 18 – 3 + 3
r = -6 (or) 3r = 18
r = -6 (or) r = 18/3
r = -6 (or) r = 6
∴ r = 6 [since, r should be a positive integer.]

18. If the coefficients of (2r + 1)th term and (r + 2)th term in the expansion of $(1 + x)^{43}$ are equal, find r.

Solution:

Given:
$(1 + x)^{43}$
We know, the coefficient of the r term in the expansion of $(1 + x)^n$ is ${}^nC_{r-1}$
So, the coefficients of the (2r + 1) and (r + 2) terms in the given expansion are ${}^{43}C_{2r+1-1}$ and ${}^{43}C_{r+2-1}$

For these coefficients to be equal, we must have
$^{43}C_{2r+1-1} = {}^{43}C_{r+2-1}$
$^{43}C_{2r} = {}^{43}C_{r+1}$
$2r = r + 1$ (or) $2r + r + 1 = 43$ [Since, $^{n}C_{r} = {}^{n}C_{s} \Rightarrow r = s$ (or) $r + s = n$]
$2r - r = 1$ (or) $3r + 1 = 43$
$r = 1$ (or) $3r = 43 – 1$
$r = 1$ (or) $3r = 42$
$r = 1$ (or) $r = 42/3$
$r = 1$ (or) $r = 14$
$\therefore r = 14$ [since, value '1' gives the same term]

19. Prove that the coefficient of (r + 1)th term in the expansion of $(1 + x)^{n+1}$ is equal to the sum of the coefficients of rth and (r + 1)th terms in the expansion of $(1 + x)^n$.
Solution:
We know, the coefficients of (r + 1)th term in $(1 + x)^{n+1}$ is $^{n+1}C_r$
So, sum of the coefficients of the rth and (r + 1)th terms in $(1 + x)^n$ is
$(1 + x)^n = {}^{n}C_{r-1} + {}^{n}C_r$
$= {}^{n+1}C_r$ [since, $^{n}C_{r+1} + {}^{n}C_r = {}^{n+1}C_{r+1}$]
Hence proved.