Exercise 1.1 Page No: 5

1. Determine whether each of the following relations are reflexive, symmetric and transitive:

(i) Relation R in the set A = {1, 2, 3,…….., 13, 14} defined as R = {(x, y) : 3x – y = 0}

(ii) Relation R in the set N of natural numbers defined as
R = {(x, y) : y = x + 5 and x < 4}

(iii) Relation R in the set A = {1, 2, 3, 4, 5, 6} as
R = {(x, y) : y is divisible by x}

(iv) Relation R in the set Z of all integers defined as
R = {(x, y) : x – y is an integer}

(v) Relation R in the set A of human beings in a town at a particular time given by

(a) R = {(x, y) : x and y work at the same place}
(b) R = {(x, y) : x and y live in the same locality}
(c) R = {(x, y) : x is exactly 7 cm taller than y}
(d) R = {(x, y) : x is wife of y}
(e) R = {(x, y) : x is father of y}

Solution:
(i)R = {(x, y) : 3x - y = 0}

A = {1, 2, 3, 4, 5, 6, ……13, 14}

Therefore, **R = {(1, 3), (2, 6), (3, 9), (4, 12)}** …(1)

As per reflexive property: (x, x)∈ R, then R is reflexive) Since there is no such pair, so R is not reflexive.

As per symmetric property: (x, y)∈ R and (y, x)∈ R, then R is symmetric. Since there is no such pair, R is not symmetric

As per transitive property: If (x, y)∈ R and (y, z)∈ R, then (x, z)∈ R. Thus R is transitive.

From (1), (1, 3) ∈ R and (3, 9) ∈ R but (1, 9) ∉ R, R is not transitive.

Therefore, R is neither reflexive, nor symmetric and nor transitive.
(ii) R = {(x, y) : y = x + 5 and x < 4} in set N of natural numbers.

Values of x are 1, 2, and 3

So, **R = {(1, 6), (2, 7), (3, 8)}**

As per reflexive property: (x, x)∈ R, then R is reflexive)

Since there is not such pair, R is not reflexive.

As per symmetric property: (x, y)∈ R and (y, x)∈ R, then R is symmetric.

Since there is no such pair, so R is not symmetric

As per transitive property: If (x, y)∈ R and (y, z)∈ R, then (x, z)∈ R. Thus R is transitive.

Since there is no such pair, so R is not transitive.

Therefore, R is neither reflexive, nor symmetric and nor transitive.

(iii) R = {(x, y) : y is divisible by x} in A = {1, 2, 3, 4, 5, 6}

From above we have,

R = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (5, 5), (6, 6)}

As per reflexive property: (x, x)∈ R, then R is reflexive.

(1, 1), (2, 2), (3, 3), (4, 4), (5, 5) and (6, 6) ∈ R . Therefore, R is reflexive.

As per symmetric property: (x, y)∈ R and (y, x)∈ R, then R is symmetric.

(1, 2) ∈ R but (2, 1) ∉R. So R is not symmetric.

As per transitive property: If (x, y)∈ R and (y, z)∈ R, then (x, z)∈ R. Thus R is transitive.

Also (1, 4) ∈ R and (4, 4) ∈ R and (1, 4) ∈ R, So R is transitive.

Therefore, R is reflexive and transitive but nor symmetric.

(iv) R = {(x, y) : x - y is an integer} in set Z of all integers.

Now, (x, x), say (1, 1) = x – y = 1 – 1 = 0 ∈ Z => R is reflexive.

(x, y) ∈ R and (y, x) ∈ R, i.e., x – y and y – x
are integers => R is symmetric.

(x, y) ∈ R and (y, z) ∈ R, then (x, z) ∈ R i.e.,

x – y and y – z and x - z are integers.

(x, z) ∈ R => R is transitive

Therefore, R is reflexive, symmetric and transitive.

(v)
(a) R = {(x, y) : x and y work at the same place}
For reflexive: x and x can work at same place
(x, x) ∈ R R
is reflexive.

For symmetric: x and y work at same place so y and x also work at same place.
(x, y) ∈ R and (y, x) ∈ R R
is symmetric.

For transitive: x and y work at same place and y and z work at same place, then x and z also work at same place.

(x, y) ∈ R and (y, z) ∈ R then (x, z) ∈ R
R is transitive

Therefore, R is reflexive, symmetric and transitive.

(b) R = {(x, y) : x and y live in the same locality}

(x, x) ∈ R => R is reflexive.

(x, y) ∈ R and (y, x) ∈ R => R is symmetric.

Again,

(x, y) ∈ R and (y, z) ∈ R then (x, z) ∈ R => R is transitive.

Therefore, R is reflexive, symmetric and transitive.
(c) R = {(x, y) : x is exactly 7 cm taller than y}

x can not be taller than x, so R is not reflexive.

x is taller than y then y can not be taller than x, so R is not symmetric.

Again, x is 7 cm taller than y and y is 7 cm taller than z, then x can not be 7 cm taller than z, so R is not transitive.

Therefore, R is neither reflexive, nor symmetric and nor transitive.

(d) R = {(x, y) : x is wife of y}

x is not wife of x, so R is not reflexive.

x is wife of y but y is not wife of x, so R is not symmetric.

Again, x is wife of y and y is wife of z then x can not be wife of z, so R is not transitive.

Therefore, R is neither reflexive, nor symmetric and nor transitive.

(e) R = {(x, y) : x is father of y}

x is not father of x, so R is not reflexive.

x is father of y but y is not father of x, so R is not symmetric.

Again, x is father of y and y is father of z then x cannot be father of z, so R is not transitive.

Therefore, R is neither reflexive, nor symmetric and nor transitive.

2. Show that the relation R in the set R of real numbers, defined as R = {(a, b) : a ≤ b^2} is neither reflexive nor symmetric nor transitive.

Solution:
R = {(a, b) : a ≤ b^2} , Relation R is defined as the set of real numbers.

(a, a) ∈ R then a ≤ a^2 , which is false. R is not reflexive.

(a, b)=(b, a) ∈ R then $a \leq b^2$ and $b \leq a^2$, it is false statement. R is not symmetric.

Now, $a \leq b^2$ and $b \leq c^2$,then $a \leq c^2$, which is false. R is not transitive

Therefore, R is neither reflexive, nor symmetric and nor transitive.

3. Check whether the relation R defined in the set {1, 2, 3, 4, 5, 6} as R = {(a, b) : b = a + 1} is reflexive, symmetric or transitive.

Solution: R = {(a, b) : b = a + 1}

R = {(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)}

When b = a, a = a + 1: which is false, So R is not reflexive.

If (a, b) = (b,a), then b = a+1 and a =b+1: Which is false, so R is not symmetric.

Now, if (a, b), (b,c) and (a, c) belongs to R then b = a+1 and c =b+1 which implies c = a + 2: Which is false, so R is not transitive.

Therefore, R is neither reflexive, nor symmetric and nor transitive.

4. Show that the relation R in R defined as R = {(a, b) : a ≤ b}, is reflexive and transitive but not symmetric.

Solution:
a ≤ a: which is true, (a, a) ∈ R, So R is reflexive.

a ≤ b but b ≤ a (false): (a, b) ∈ R but (b, a) ∉ R, So R is not symmetric.

Again, a ≤ b and b ≤ c then a ≤ c : (a, b) ∈ R and (b, c) and (a, c) ∈ R, So R is transitive.

Therefore, R is reflexive and transitive but not symmetric.

5. Check whether the relation R in R defined by R = {(a, b) : $a \leq b^3$} is reflexive, symmetric or transitive.

Solution: R = {(a, b) : $a \leq b^3$}

$a \leq a^3$: which is true, (a, a) ∉ R, So R is not reflexive.

$a \leq b^3$ but $b \leq a^3$ (false): (a, b) ∈ R but (b, a) ∉ R, So R is not symmetric.

Again, $a \leq b^3$ and $b \leq c^3$ then $a \leq c^3$ (false) : (a, b) ∈ R and (b, c) ∈ R and (a, c) ∉ R, So R is transitive.

Therefore, R is neither reflexive, nor transitive and nor symmetric.

6. Show that the relation R in the set {1, 2, 3} given by R = {(1, 2), (2, 1)} is symmetric but neither reflexive nor transitive.

Solution:

R = {(1, 2), (2, 1)}

(x, x) ∉ R. R is not reflexive.

(1, 2) ∈ R and (2,1) ∈ R. R is symmetric.

Again, (x, y) ∈ R and (y, z) ∈ R then (x, z) does not imply to R. R is not transitive.

Therefore, R is symmetric but neither reflexive nor transitive.

7. Show that the relation R in the set A of all the books in a library of a college, given by R = {(x, y) : x and y have same number of pages} is an equivalence relation.

Solution:
Books x and x have same number of pages. (x, x) ∈ R. R is reflexive.

If (x, y) ∈ R and (y, x) ∈ R, so R is symmetric.
Because, Books x and y have same number of pages and Books y and x have same number of pages.

Again, (x, y) ∈ R and (y, z) ∈ R and (x, z) ∈ R. R is transitive.

Therefore, R is an equivalence relation.

8. Show that the relation R in the set A = {1, 2, 3, 4, 5} given by R = {(a, b) : |a – b| is even}, is an equivalence relation. Show that all the elements of {1, 3, 5} are related to each other and all the elements of {2, 4} are related to each other. But no element of {1, 3, 5} is related to any element of {2, 4}.

Solution:

A = {1, 2, 3, 4, 5} and R = {(a, b) : |a – b| is even}

We get, R = {(1, 3), (1, 5), (3, 5), (2, 4)}

For (a, a), |a – b| = |a – a| = 0 is even. Therfore, R is reflexive.

If |a – b| is even, then |b – a| is also even. R is symmetric.

Again, if |a – b| and |b – c| is even then |a – c| is also even. R is transitive.

Therefore, R is an equivalence relation.

(b) We have to show that, Elements of {1, 3, 5} are related to each other.

|1 – 3| = 2
|3 – 5| = 2 |1
- 5| = 4
All are even numbers.

Elements of {1, 3, 5} are related to each other.

Similarly, |2 - 4| = 2 (even number), elements of (2, 4) are related to each other.

Hence no element of {1, 3, 5} is related to any element of {2, 4}.

9. Show that each of the relation R in the set A = {x ∈ Z : 0 ≤ x ≤ 12}, given by (i) R = {(a, b) : |a – b| is a multiple of 4}
(ii) R = {(a, b) : a = b} is an equivalence relation. Find the set of all elements related to 1 in each case.

Solution:

(i) A = {x ∈ Z : 0 ≤ x ≤ 12} So, A = {0, 1, 2, 3, ………, 12}

Now R = {(a, b) : |a – b| is a multiple of 4}

R = {(4, 0), (0, 4), (5, 1), (1, 5), (6, 2), (2, 6), ….., (12, 9), (9, 12),…., (8, 0), (0, 8), …., (8, 4), (4, 8),….., (12, 12)}

Here, (x, x) = |4-4| = |8-8|= |12-12| = 0 : multiple of 4.

R is reflexive.

|a – b| and |b – a| are multiple of 4. (a, b) ∈ R and (b, a) ∈ R.

R is symmetric.

And |a – b| and |b – c| then |a - c| are multiple of 4. (a, b) ∈ R and (b, c) ∈ R and (a, c) ∈ R R is transitive.

Hence R is an equivalence relation.
(ii) Here, (a, a) = a = a.

(a, a) ∈ R . So R is reflexive.

a = b and b = a. (a, b) ∈ R and (b, a) ∈ R.

R is symmetric.

And a = b and b = c then a = c. (a, b) ∈ R and (b, c) ∈ R and (a, c) ∈ R R is transitive.

Hence R is an equivalence relation.

Now set of all elements related to 1 in each case is

(i) Required set = {1, 5, 9}
(ii) Required set = {1}

10. Give an example of a relation. Which is (i) Symmetric but neither reflexive nor transitive.
(ii) Transitive but neither reflexive nor symmetric.
(iii)Reflexive and symmetric but not transitive. (iv) Reflexive and transitive but not symmetric.
(v) Symmetric and transitive but not reflexive.

Solution:
(i)Consider a relation R = {(1, 2), (2, 1)} in the set {1, 2, 3}

(x, x) ∉ R. R is not reflexive.

(1, 2) ∈ R and (2,1) ∈ R. R is symmetric.

Again, (x, y) ∈ R and (y, z) ∈ R then (x, z) does not imply to R. R is not transitive.

Therefore, R is symmetric but neither reflexive nor transitive.

(ii) Relation R = {(a, b): a > b}

a > a (false statement).
Also a > b but b > a (false statement) and
If a > b but b > c, this implies a > c

Therefore, R is transitive, but neither reflexive nor symmetric.

(iii) R = {a, b): a is friend of b}

a is friend of a. R is reflexive.

Also a is friend of b and b is friend of a. R is symmetric.

Also if a is friend of b and b is friend of c then a cannot be friend of c. R is not transitive.

Therefore, R is reflexive and symmetric but not transitive.

(iv) Say R is defined in R as R = {(a, b) : a ≤ b}

a ≤ a: which is true, (a, a) ∈ R, So R is reflexive.

a ≤ b but b ≤ a (false): (a, b) ∈ R but (b, a) ∉ R, So R is not symmetric.

Again, a ≤ b and b ≤ c then a ≤ c : (a, b) ∈ R and (b, c) and (a, c) ∈ R, So R is transitive.

Therefore, R is reflexive and transitive but not symmetric.

(v)R = {(a, b): a is sister of b} (suppose a and b are female)

a is not sister of a. R is not reflexive.

a is sister of b and b is sister of a. R is symmetric.

Again, a is sister of b and b is sister of c then a is sister of c.

Therefore, R is symmetric and transitive but not reflexive.

11. Show that the relation R in the set A of points in a plane given by R = {(P, Q) : distance of the point P from the origin is same as the distance of the point Q from the origin}, is an equivalence relation. Further, show that the set of all points related to a point P ≠ (0, 0) is the circle passing through P with origin as centre.

Solution: R = {(P, Q): distance of the point P from the origin is the same as the distance of the point Q from the origin}

Say "O" is origin Point.
Since the distance of the point P from the origin is always the same as the distance of the same point P from the origin.
OP = OP

So (P, P) R. R is reflexive.

Distance of the point P from the origin is the same as the distance of the point Q from the origin

OP = OQ then OQ = OP
R is symmetric.

Also OP = OQ and OQ = OR then OP = OR. R is transitive.

Therefore, R is an equivalent relation.

12. Show that the relation R defined in the set A of all triangles as R = {(T_1, T_2) : T_1 is similar to T_2}, is equivalence relation. Consider three right angle triangles T_1 with sides 3, 4, 5, T_2 with sides 5, 12, 13 and T_3 with sides 6, 8, 10. Which triangles among T_1, T_2 and T_3 are related?

Solution:
Case I:
T_1, T_2 are triangle.

R = {(T_1, T_2): T_1 is similar to T_2}

Check for reflexive:

As We know that each triangle is similar to itself, so (T_1, T_1) ∈ R R
is reflexive.

Check for symmetric:

Also two triangles are similar, then T_1 is similar to T_2 and T_2 is similar to T_1, so $(T_1, T_2) \in R$ and $(T_2, T_1) \in R$ R
is symmetric.

Check for transitive:

Again, if then T_1 is similar to T_2 and T_2 is similar to T_3, then T_1 is similar to T_3 , so $(T_1, T_2) \in R$ and $(T_2, T_3) \in R$ and $(T_1, T_3) \in R$
R is transitive

Therefore, R is an equivalent relation.

Case 2: It is given that T_1, T_2 and T_3 are right angled triangles.

T_1 with sides 3, 4, 5
T_2 with sides 5, 12, 13 and
T_3 with sides 6, 8, 10

Since, two triangles are similar if corresponding sides are proportional.

Therefore, $3/6 = 4/8 = 5/10 = 1/2$

Therefore, T_1 and T_3 are related.

13. Show that the relation R defined in the set A of all polygons as R = {(P_1, P_2) :P_1 and P_2 have same number of sides}, is an equivalence relation. What is the set of all elements in A related to the right angle triangle T with sides 3, 4 and 5?

Solution:

Case I:
R = {(P_1, P_2) :P_1 and P_2 have same number of sides} **Check for reflexive:**

P_1 and P_1 have same number of sides, So R is reflexive.

Check for symmetric:

P_1 and P_2 have same number of sides then P_2 and P_1 have same number of sides, so $(P_1, P_2) \in R$ and $(P_2, P_1) \in R$ R
is symmetric.

Check for transitive:

Again, P_1 and P_2 have same number of sides, and P_2 and P_3 have same number of sides, then also P_1 and P_3 have same number of sides .
So $(P_1, P_2) \in R$ and $(P_2, P_3) \in R$ and $(P_1, P_3) \in R$
R is transitive

Therefore, R is an equivalent relation.

Since 3, 4, 5 are the sides of a triangle, the triangle is right angled triangle. Therefore, the set A is the set of right angled triangle.

14. Let L be the set of all lines in XY plane and R be the relation in L defined as R = $\{(L_1, L_2) : L_1$ is parallel to $L_2\}$. Show that R is an equivalence relation. Find the set of all lines related to the line y = 2x + 4.

Solution:

L_1 is parallel to itself i.e., $(L_1, L_1) \in R$

R is reflexive

Now, let $(L_1, L_2) \in R$

L_1 is parallel to L_2 and L_2 is parallel to L_1

$(L_2, L_1) \in R$, Therefore, R is symmetric

Now, let (L_1, L_2), $(L_2, L_3) \in R$

L_1 is parallel to L_2. Also, L_2 is parallel to L_3

L_1 is parallel to L_3

Therefore, R is transitive

Hence, R is an equivalence relation.

Again, The set of all lines related to the line y = 2x + 4, is the set of all its parallel lines.
Slope of given line is m = 2.
As we know slope of all parallel lines are same.
Hence, the set of all related to y = 2x + 4 is y = 2x + k, where $k \in R$.

15. Let R be the relation in the set {1, 2, 3, 4} given by R = {(1, 2), (2, 2), (1, 1), (4,4), (1, 3), (3, 3), (3, 2)}. Choose the correct answer.
(A) R is reflexive and symmetric but not transitive.
(B) R is reflexive and transitive but not symmetric.

(C) R is symmetric and transitive but not reflexive.
(D) R is an equivalence relation.

Solution:
Let R be the relation in the set {1, 2, 3, 4} given by R = {(1, 2), (2, 2), (1, 1), (4,4), (1, 3), (3, 3), (3, 2)}.

Step 1: (1, 1), (2, 2), (3, 3), (4, 4) ∈ R R.   R is reflexive.

Step 2: (1, 2) ∈ R but (2, 1) ∉ R. R is not symmetric.

Step 3: Consider any set of points, (1, 3) ∈ R and (3, 2) ∈ R then (1, 2) ∈ R. So R is transitive.

Option (B) is correct.

16. Let R be the relation in the set N given by R = {(a, b) : a = b – 2, b > 6}. Choose the correct answer.
(A) (2, 4) ∈ R (B) (3, 8) ∈ R (C) (6, 8) ∈ R (D) (8, 7) ∈ R

Solution: R = {(a, b) : a = b – 2, b > 6}

(A) Incorrect : Value of b = 4, not true.

(B) Incorrect : a = 3 and b = 8 > 6 a = b – 2 =>
3 = 8 – 2 and 3 = 6, which is false.

(C) Correct: a = 6 and b = 8 > 6 a = b – 2 => 6
= 8 – 2 and 6 = 6, which is true.

(D) Incorrect : a = 8 and b = 7 > 6 a = b – 2 =>
8 = 7 – 2 and 8 = 5, which is false.

Therefore, option (C) is correct.

Exercise 1.2 Page No: 10

1. Show that the function f : R → R defined by f(x) = 1/x is one-one and onto, where R is the set of all non-zero real numbers. Is the result true, if the domain R_* is replaced by N with co-domain being same as R ?

Solution:

Given: f : R → R defined by f(x) = 1/x

Check for One-One

$f(x_1) = \frac{1}{x_1}$ and $f(x_2) = \frac{1}{x_2}$

$If\ f(x_1) = f(x_2)\ then\ \frac{1}{x_1} = \frac{1}{x_2}$

This implies $x_1 = x_2$

Therefore, f is one-one function.

Check for onto

f(x) = 1/x or
y = 1/x or x
= 1/y f(1/y)
= y
Therefore, f is onto function.

Again, If $f(x_1) = f(x_2)$

Say, $n_1, n_2 \in R$

$$\frac{1}{n_1} = \frac{1}{n_2}$$

So $n_1 = n_2$
Therefore, f is one-one

Every real number belonging to co-domain may not have a pre-image in N. for example, 1/3 and 3/2 are not belongs N. So N is not onto.

2. Check the injectivity and surjectivity of the following functions:

(i) f : N → N given by f(x) = x^2

(ii) f : Z → Z given by f(x) = x^2

(iii) f : R → R given by f(x) = x^2

(iv) f : N → N given by f(x) = x^3

(v) f : Z → Z given by f(x) = x^3

Solution:

(i) f : N → N given by f(x) = x^2

For x, y ∈ N => f(x) = f(y) which implies $x^2 = y^2$

⇨ x = y

Therefore f is injective.

There are such numbers of co-domain which have no image in domain N.

Say, 3 ∈ N, but there is no pre-image in domain of f. such that f(x) = x^2 = 3.

f is not surjective.

Therefore, f is injective but not surjective.

(ii) Given, f : Z → Z given by f(x) = x^2

Here, Z = {0, ±1, ±2, ±3, ±4, ...}

f(-1) = f(1) = 1

But -1 not equal to 1.

f is not injective.

There are many numbers of co-domain which have no image in domain Z.

For example, -3 ∈ co-domain Z, but -3 ∉ domain Z f is not surjective.

Therefore, f is neither injective nor surjective.

(iii) f : R → R given by f(x) = x^2

f(-1) = f(1) = 1

But -1 not equal to 1.

f is not injective.

There are many numbers of co-domain which have no image in domain R.

For example, -3 ∈ co-domain R, but there does not exist any x in domain R where x^2 = -3
f is not surjective.

Therefore, f is neither injective nor surjective.

(iv) f : N → N given by f(x) = x^3

For x, y ∈N => f(x) = f(y) which implies $x^3 = y^3$ ⇨
x = y
Therefore f is injective.

There are many numbers of co-domain which have no image in domain N.

For example, 4 ∈ co-domain N, but there does not exist any x in domain N where x^3 = 4. f is not surjective.

Therefore, f is injective but not surjective.

(v) f : Z → Z given by f(x) = x^3

For x, y ∈Z => f(x) = f(y) which implies $x^3 = y^3$ ⇨
x = y

Therefore f is injective.

There are many numbers of co-domain which have no image in domain Z.

For example, 4 ∈ co-domain N, but there does not exist any x in domain Z where $x^3 = 4$. f is not surjective.

Therefore, f is injective but not surjective.

3. Prove that the Greatest Integer Function f : R → R, given by f(x) = [x], is neither oneone nor onto, where [x] denotes the greatest integer less than or equal to x.

Solution:

Function f : R → R, given by f(x) = [x] f(x)
= 1, because 1 ≤ x ≤ 2

f(1.2) = [1.2] = 1
f(1.9) = [1.9] = 1 But
1.2 ≠ 1.9

f is not one-one.

There is no fraction proper or improper belonging to co-domain of f has any pre-image in its domain.

For example, f(x) = [x] is always an integer

for 0.7 belongs to R there does not exist any x in domain R where f(x) = 0.7 f is not onto.

Hence proved, the Greatest Integer Function is neither one-one nor onto.

4. Show that the Modulus Function f : R → R, given by f(x) = | x |, is neither one-one nor onto, where | x | is x, if x is positive or 0 and |x | is – x, if x is negative.

Solution:

f : R → R, given by f(x) = | x |, defined as

$$f(x) = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

f contains values like (-1, 1),(1, 1),(-2, 2)(2,2)

f(-1) = f(1), but -1 1

f is not one-one.

R contains some negative numbers which are not images of any real number since f(x) = |x| is always non-negative. So f is not onto.

Hence, Modulus Function is neither one-one nor onto.

5. Show that the Signum Function f : R → R, given by

$$f(x) = \begin{cases} 1, \text{if } x > 0 \\ 0, \text{if } x = 0 \\ 1, \text{if } x < 0 \end{cases}$$

is neither one-one nor onto.

Solution: Signum Function f : R → R, given by

$$f(x) = \begin{cases} 1, \text{if } x > 0 \\ 0, \text{if } x = 0 \\ 1, \text{if } x < 0 \end{cases}$$

f(1) = f(2) = 1

This implies, for n > 0, $f(x_1) = f(x_2) = 1$

$x_1 \neq x_2$

f is not one-one.

f(x) has only 3 values, (-1, 0 1). Other than these 3 values of co-domain R has no any preimage its domain.

f is not onto.

Hence, Signum Function is neither one-one nor onto.

6. Let A = {1, 2, 3}, B = {4, 5, 6, 7} and let f = {(1, 4), (2, 5), (3, 6)} be a function from A to B. Show that f is one-one.

Solution:
A = {1, 2, 3}
B = {4, 5, 6, 7} and f = {(1, 4), (2, 5), (3, 6)}

f(1) = 4, f(2) = 5 and f(3) = 6

Here, also distinct elements of A have distinct images in B.

Therefore, f is one-one.

7. In each of the following cases, state whether the function is one-one, onto or bijective. Justify your answer.
(i) f : R → R defined by f(x) = 3 – 4x (ii) f : R → R defined by f(x) = 1 + x^2

Solution:
(i) f : R → R defined by f(x) = 3 – 4x
If x_1, x_2 ∈R then

$f(x_1) = 3 – 4x_1$ and
$f(x_2) = 3 – 4x_2$

If $f(x_1) = f(x_2)$ then $x_1 = x_2$

Therefore, f is one-one.

Again, f(x) = 3 –
4x or y = 3 – 4x or
x = (3-y)/4 in R

f((3-y)/4) = 3 – 4((3-y)/4) = y

f is onto.

Hence f is onto or bijective.

(ii) f : R → R defined by f(x) = 1 + x^2

If x_1, x_2 ∈R then

$f(x_1) = 1 + x_1^2$ and $f(x_2) = 1 + x_2^2$

If $f(x_1) = f(x_2)$ then $x_1^2 = x_2^2$

This implies $x_1 \neq x_2$

Therefore, f is not one-one

Again, if every element of co-domain is image of some element of Domain under f, such that f(x) = y

f(x) = 1 + x^2

y = f(x) = 1 + x^2

or x = $\pm\sqrt{1-y}$

Therefore, $f(\sqrt{1-y}) = 2 - y \neq y$

Therefore, f is not onto or bijective.

8. Let A and B be sets. Show that f : A × B → B × A such that f(a, b) = (b, a) is bijective function.

Solution:

Step 1: Check for Injectivity:

Let (a_1, b_1) and (a_2, b_2) ∈ A x B such that

$f(a_1, b_1) = (a_2, b_2)$

This implies, (b_1, a_1) and (b_2, a_2)

$b_1 = b_2$ and $a_1 = a_2$

$(a_1, b_1) = (a_2, b_2)$ for all (a_1, b_1) and $(a_2, b_2) \in A \times B$

Therefore, f is injective.

Step 2: Check for Surjectivity:

Let (b, a) be any element of B x A. Then $a \in A$ and $b \in B$

This implies $(a, b) \in A \times B$

For all $(b, a) \in B \times A$, their exists $(a, b) \in A \times B$

Therefore, f: A x B -> B x A is bijective function.

9. Let f : N → N be defined by

$$f(n) = \begin{cases} \frac{n+1}{2}, & \text{if } n \text{ is odd} \\ \frac{n}{2}, & \text{if } n \text{ is even} \end{cases} \quad \text{for all } n \in \mathbf{N}$$

State whether the function f is bijective. Justify your answer

Solution:

$$f(n) = \begin{cases} \frac{n+1}{2}, & \text{if } n \text{ is odd} \\ \frac{n}{2}, & \text{if } n \text{ is even} \end{cases} \quad \text{for all } n \in \mathbf{N}$$

For n = 1, 2

f(1) = (n+1)/2 = (1+1)/2 = 1 and

f(2) = (n)/2 = (2)/2 = 1

f(1) = f(2), but 1 ≠ 2

f is not one-one.

For a natural number, “a” in co-domain N

If n is odd

n = 2k + 1 for k ∈ N , then 4k + 1 ∈ N such that

f(4k+1) = (4k+1+1)/2 = 2k + 1

If n is even

n= 2k for some k ∈ N such that
f(4k) = 4k/2 = 2k
f is onto

Therefore, f is onto but not bijective function.

10. Let A = R – {3} and B = R – {1}. Consider the function f : A → B defined by f(x) = (x-2)/(x-3)
Is f one-one and onto? Justify your answer.

Solution: A = R – {3} and B = R – {1}

f : A → B defined by f(x) = (x-2)/(x-3)

Let (x, y) ∈A then

$$f(x) = \frac{x - 2}{x - 3} \; and \; f(y) = \frac{y - 2}{y - 3}$$

For f(x) = f(y)

$$\frac{x-2}{x-3}=\frac{y-2}{y-3}$$
$$(x-2)(y-3)=(y-2)(x-3)$$
$$xy-3x-2y+6=xy-3y-2x+6$$
$$-3x-2y=-3y-2x$$
$$-3x+2x=-3y+2y$$
$$-x=-y$$
$$x=y$$

Again, f(x) = (x-2)/(x-3)
or y = f(x) = (x-2)/(x-3)
y = (x-2)/(x-3) y(x – 3) =
x – 2 xy – 3y = x – 2
x(y – 1) = 3y – 2

or x = (3y-2)/(y-1)

Now, f((3y-2)/(y-1)) = $\dfrac{\frac{3y-2}{y-1}-2}{\frac{3y-2}{y-1}-3}$ = y

f(x) = y

Therefore, f is onto function.

11. Let f : R → R be defined as f(x) = x^4. Choose the correct answer.
(A) f is one-one onto (B) f is many-one onto
(C) f is one-one but not onto (D) f is neither one-one nor onto.

Solution:
f : R → R be defined as f(x) = x^4

let x and y belongs to R such that, f(x) = f(y)

$x^4 = y^4$ or x = ± y

f is not one-one function.

Now, $y = f(x) = x^4$ Or $x = \pm y^{1/4}$

$f(y^{1/4}) = y$ and $f(-y^{1/4}) = -y$

Therefore, f is not onto function.

Option D is correct.

12. Let f : R → R be defined as f(x) = 3x. Choose the correct answer.
(A) f is one-one onto (B) f is many-one onto
(C) f is one-one but not onto (D) f is neither one-one nor onto.

Solution: f : R → R be defined as f(x) = 3x

let x and y belongs to R such that f(x) = f(y)

3x = 3y or x = y

f is one-one function.

Now, y = f(x) = 3x

Or x = y/3

f(x) = f(y/3) = y

Therefore, f is onto function.

Option (A) is correct.

Exercise 1.3

1. Let f : {1, 3, 4} → {1, 2, 5} and g : {1, 2, 5} → {1, 3} be given by f = {(1, 2), (3, 5), (4, 1)} and g = {(1, 3), (2, 3), (5, 1)}. Write down *gof*.

Solution:

Given function, f : {1, 3, 4} → {1, 2, 5} and g : {1, 2, 5} → {1, 3} be given by

f = {(1, 2), (3, 5), (4, 1)} and g = {(1, 3), (2, 3), (5, 1)}

Find *gof*.

At f(1) = 2 and g(2) = 3, *gof* is

gof(1) = g(f(1)) = g(2) = 3

At f(3) = 5 and g(5) = 1, *gof* is

gof(3) = g(f(3)) = g(5) = 1

At f(4) = 1 and g(1) = 3, *gof* is

gof(4) = g(f(4)) = g(1) = 3

Therefore, *gof* = {(1,3), (3,1), (4,3)}

2. Let f, g and h be functions from R to R. Show that
(f + g) oh = foh + goh
(f . g) oh = (foh) . (goh)

Solution:

LHS = *(f + g) oh*

= *(f+g)(h(x))*

= *f(h(x)) + g(h(x))*

= *foh + goh*

= RHS

Again,

LHS = *(f . g) oh*

= *f.g(h(x))*

= *f(h(x)) . g(h(x))*

= *(foh) . (goh)*

= RHS

3. Find gof and fog, if

(i) f(x) = | x | and g(x) = | 5x – 2 |

(ii) f(x) = 8x^3 and g(x) = x$^{1/3}$.

Solution:

(i) f(x) = | x | and g(x) = | 5x – 2 |

gof =(gof)(x) = g(f(x) = g(|x|) = |5 |x| - 2|

fog = (fog)(x) = f(g(x)) = f(| 5x – 2 |) = ||5x – 2|| = | 5x – 2 |

(ii) f(x) = 8x^3 and g(x) = x$^{1/3}$ **.**

gof =(gof)(x) = g(f(x) = g(8x^3) = (8x^3)$^{1/3}$ = 2x

fog = (fog)(x) = f(g(x)) = f(x$^{1/3}$) = 8(x$^{1/3}$)^3 = 8x

4. If f(x) = $\frac{(4x+3)}{(6x-4)}$, x ≠ 2/3, Show that *fof*(x) = x, for all x ≠ 2/3. What is the inverse of f.

Solution:

f(x) = $\frac{(4x+3)}{(6x-4)}$, x ≠ 2/3,

$$= \frac{4\left(\frac{4x+3}{6x-4}\right)+3}{6\left(\frac{4x+3}{6x-4}\right)-4}$$

$$= \frac{16x+12+18x-12}{24x+18-24x+16}$$

$$= \frac{34x}{34}$$

$$= x$$

Therefore, *fof*(x) = x for all x ≠ 2/3.

Again, fof = I

The inverse of the given function, f is f.

5. State with reason whether following functions have inverse
(i) f : {1, 2, 3, 4} → {10} with f = {(1, 10), (2, 10), (3, 10), (4, 10)}

(ii) g : {5, 6, 7, 8} → {1, 2, 3, 4} with g = {(5, 4), (6, 3), (7, 4), (8, 2)}

(iii) h : {2, 3, 4, 5} → {7, 9, 11, 13} with h = {(2, 7), (3, 9), (4, 11), (5, 13)}

Solution:

(i) f : {1, 2, 3, 4} → {10} with f = {(1, 10), (2, 10), (3, 10), (4, 10)}

f has many-one function like f(1) = f(2) = f(3) = f(4) = 10, therefore f has no inverse.

(ii) g : {5, 6, 7, 8} → {1, 2, 3, 4} with g = {(5, 4), (6, 3), (7, 4), (8, 2)}

g has many-one function like g(5) = g(7) = 4, therefore g has no inverse.

(iii) h : {2, 3, 4, 5} → {7, 9, 11, 13} with h = {(2, 7), (3, 9), (4, 11), (5, 13)}

All elements have different images under h. So h is one-one onto function, therefore, h has an inverse.

6. Show that f : [–1, 1] → R, given by f (x) = x/(x+2) is one-one. Find the inverse of the function f : [–1, 1] → Range f.

(Hint: For y ∈ Range f, y = f(x) = x/(x+2), for some x in [–1, 1], i.e., x = 2y/(1-y).

Solution:
Given function: (x) = x/(x+2)
Let x, y ∈ [–1, 1]
Let f(x) = f(y)

x/(x+2) = y/(y+2)
xy + 2x = xy + 2y
x = y f is one-
one.

Again,

Since f : [–1, 1] → Range f is onto

say, y = x/(x+2)

yx + 2y = x

x(1 - y) = 2y

or x = 2y/(1-y)

$x = f^{-1}(y) = 2y/(1-y)$; y not equal to 1

f is onto function, and $f^{-1}(x) = 2x/(1-x)$.

7. Consider f : R → R given by f(x) = 4x + 3. Show that f is invertible. Find the inverse of f.
Solution:
Consider f : R → R given by f(x) = 4x + 3

Say, x, y ∈ R

Let f(x) = f(y) then 4x
+ 3 = 4y + 3
x = y f is one-one function.

Let y ∈ Range of f

y = 4x + 3

or x = (y-3)/4

Here, f((y-3)/4) = 4((y-3)/4) + 3 = y

This implies f(x) = y

So f is onto

Therefore, f is invertible.

Inverse of f is x = f^{-1} (y) = (y-3)/4.

8. Consider f : $R_+ \to [4, \infty)$ given by $f(x) = x^2 + 4$. Show that f is invertible with the inverse f^{-1} of f given by $f^{-1}(y) = \sqrt{y-4}$, where R_+ is the set of all non-negative real numbers.

Solution:
Consider f : $R_+ \to [4, \infty)$ given by $f(x) = x^2 + 4$

Let x, y ∈ R → [4, ∞) then

f(x) = $x^2 + 4$ and

f(y) = $y^2 + 4$

if f(x) = f(y) then $x^2 + 4 = y^2 + 4$

or x = y

f is one-one.

Now $y = f(x) = x^2 + 4$ or $x = \sqrt{y-4}$ as $x > 0$

$f(\sqrt{y-4}) = (\sqrt{y-4})^2 + 4 = y$

$f(x) = y$

f is onto function.

Therefore, f is invertible and Inverse of f is $f^{-1}(y) = \sqrt{y-4}$.

9. Consider $f : R_+ \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$. Show that f is invertible with

$$f^{-1}(y) = \left(\frac{(\sqrt{y+6})-1}{3}\right)$$

Solution:

Consider $f : R_+ \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$

Consider $f : R_+ \to [4, \infty)$ given by $f(x) = x^2 + 4$

Let $x, y \in R \to [-5, \infty)$ then

$f(x) = 9x^2 + 6x - 5$ and

$f(y) = 9y^2 + 6y - 5$

if $f(x) = f(y)$ then $9x^2 + 6x - 5 = 9y^2 + 6y - 5$

$9(x^2 - y^2) + 6(x - y) = 0$

$9\{(x-y)(x+y)\} + 6(x - y) = 0$

$(x - y)(9)(x+y) + 6) = 0$

either $x - y = 0$ or $9(x+y) + 6 = 0$

Say $x - y = 0$, then $x = y$. So f is one-one.

Now, $y = f(x) = 9x^2 + 6x - 5$

Solving this quadratic equation, we have x = $\frac{-6\pm6\sqrt{y+6}}{18}$ $or\ x = \frac{\sqrt{y+6}-1}{3}$

So, f(x) = f($\frac{\sqrt{y+6}-1}{3}$) = 9($\frac{\sqrt{y+6}-1}{3}$)^2 + 6($\frac{\sqrt{y+6}-1}{3}$) – 5

= y + 7 – $2\sqrt{y+6}$ + 2 $\sqrt{y+6}$ – 2 – 5= y
f(x) = y, therefore, f is onto. f(x) is
invertible and $f^{-1}(x) = \frac{\sqrt{y+6}-1}{3}$.

10. Let f : X → Y be an invertible function. Show that f has unique inverse.

(Hint: suppose g_1 and g_2 are two inverses of f. Then for all y ∈ Y,*fog*$_1$(y) = 1_Y(y) = *fog*$_2$(y). Use one-one ness of f)

Solution:
Given, f : X → Y be an invertible function. And g_1 and g_2 are two inverses of f.

For all y ∈ Y, we get

$fog_1(y) = 1_Y(y) = fog_2(y)$

$f(g_1(y)) = f(g_2(y))$

$g_1(y) = g_2(y)$

$g_1 = g_2$

Hence f has unique inverse.

11. Consider f : {1, 2, 3} → {a, b, c} given by f(1) = a, f(2) = b and f(3) = c. Find f^{-1} and show that $(f^{-1})^{-1}$ = f.

Solution:

Consider f : {1, 2, 3} → {a, b, c} given by f(1) = a, f(2) = b and f(3) = c

So f = {(a, 1), (b, 2), (c, 3)}

Hence f^{-1} (a) = 1, f^{-1} (b) = 2 and f^{-1} (c) = 3

Now, f^{-1} = {(a, 1), (b, 2), (c, 3)}

Therefore, inverse of f^{-1} = $(f^{-1})^{-1}$ = {(1, a), (2, b), (3, c)} = f

Hence $(f^{-1})^{-1}$ = f.

13. If f: R → R be given by f(x) =$(3 - x^3)^{\frac{1}{3}}$, then *fof*(x) is

(A) $x^{1/3}$ (B) x^3 (C) x (D) $(3 - x^3)$

Solution:

f: R → R be given by f(x) =$(3 - x^3)^{\frac{1}{3}}$, then

fof(x) = f(f(x))

$$= f\left(\left(3-x^3\right)^{\frac{1}{3}}\right)$$

$$= \left[3-\left(\left(3-x^3\right)^{\frac{1}{3}}\right)^3\right]^{\frac{1}{3}}$$

$$= \left[3-\left(3-x^3\right)\right]^{\frac{1}{3}}$$

$$= \left(x^3\right)^{\frac{1}{3}} = x$$

Option (C) is correct.

14. Let f : R – { -4/3 } → R be a function defined as f(x) =$\frac{4x}{3x+4}$. The inverse of f is the map g : Range f → R – { -4/3 } given by

(A) g(y) = 3y/(3-4y) (B) g(y) = 4y/(4-3y)

(C) g(y) = 4y/(3-4y) (D) g(y) = 3y/(4-3y)

Solution:

Let f : R – { -4/3 } → R be a function defined as f(x) = $\frac{4x}{3x+4}$. And Range f → R – { -4/3 }

y = f(x) = $\frac{4x}{3x+4}$

y(3x + 4) = 4x

3xy + 4y = 4x

x(3y – 4) = -4y

x = 4y/(4-3y)

Therefore, f^{-1} (y) = g(y) = 4y/(4-3y). Option (B) is the correct answer.

EDUGROSS
WISDOMISING KNOWLEDGE

Exercise 1.4

1. Determine whether or not each of the definition of ∗ given below gives a binary operation. In the event that ∗ is not a binary operation, give justification for this.

(i) On Z^+, define ∗ by a ∗ b = a – b

(ii) On Z^+, define ∗ by a ∗ b = ab

(iii) On R, define ∗ by a ∗ b = ab^2

(iv) On Z^+, define ∗ by a ∗ b = | a – b |

(v) On Z+, define ∗ by a ∗ b = a

Solution:

(i) On Z^+, define ∗ by a ∗ b = a – b

On Z^+ = {1, 2,3 , 4, 5,…….}

Let a = 1 and b = 2

Therefore, a ∗ b = a – b = 1 – 2 = -1 $\notin Z^+$

operation * is not a binary operation on Z^+ .

(ii) On Z^+, define ∗ by a ∗ b = ab

On Z^+ = {1, 2,3 , 4, 5,…….}

Let a = 2 and b = 3

Therefore, a ∗ b = a b = 2 * 3 = 6 $\in Z^+$

operation * is a binary operation on Z^+

(iii) On R, define ∗ by a ∗ b = ab^2

R = { - ∞, ……, -1, 0, 1, 2,……, ∞}

Let a = 1.2 and b = 2

Therefore, $a * b = ab^2 = (1.2) \times 2^2 = 4.8 \in R$

Operation * is a binary operation on R.

(iv) On Z^+, define $*$ by $a * b = | a – b |$

On Z^+ = {1, 2,3 , 4, 5,…….}

Let a = 2 and b = 3

Therefore, $a * b = a b = 2 * 3 = 6 \in Z^+$

operation * is a binary operation on Z^+

(v) On Z+, define $*$ by $a * b = a$

On Z^+ = {1, 2, 3, 4, 5,…….}

Let a = 2 and b = 1

Therefore, $a * b = 2 \in Z^+$

Operation * is a binary operation on Z^+ .

2. For each operation $*$ defined below, determine whether $*$ is binary, commutative or associative.

(i) On Z, define $a * b = a – b$

(ii) On Q, define $a * b = ab + 1$

(iii) On Q, define $a * b = ab/2$

(iv) On Z^+, define $a * b = 2^{ab}$

(v) On Z^+, define $a * b = a^b$

(vi) On R – {– 1}, define a ∗ b = a/(b+1)

Solution:

(i) On Z, define a ∗ b = a – b

Step 1: Check for commutative

Consider ∗ is commutative, then

a ∗ b = b * a

Which means, a – b = b – a (not true)

Therefore, ∗ is not commutative.

Step 2: Check for Associative.

Consider ∗ is associative, then

(a ∗ b)* c = a * (b * c)

LHS = (a ∗ b)* c = (a – b) * c

= a – b – c

RHS = a * (b * c) = a - (b- c)

= a – (b – c)

= a – b + c

This implies LHS ≠ RHS

Therefore, ∗ is not associative.

(ii) On Q, define a ∗ b = ab + 1

Step 1: Check for commutative

Consider ∗ is commutative, then

a ∗ b = b * a

Which means, ab + 1 = ba + 1

or ab + 1 = ab + 1 (which is true)

a ∗ b = b * a for all a, b ∈Q

Therefore, ∗is commutative.

Step 2: Check for Associative.

Consider ∗ is associative, then

(a ∗ b)* c = a * (b * c)

LHS = (a ∗ b) * c = (ab + 1) * c

= (ab + 1)c + 1

= abc + c + 1

RHS = a * (b * c) = a * (bc + 1)

= a(bc + 1) + 1

= abc + a + 1

This implies LHS ≠ RHS

Therefore, ∗is not associative.

(iii) On Q, define a ∗ b = ab/2

Step 1: Check for commutative

Consider ∗ is commutative, then

a ∗ b = b * a

Which means, ab/2 = ba/2

or ab/2 = ab/2 (which is true)

a ∗ b = b * a for all a, b ∈Q

Therefore, ∗is commutative.

Step 2: Check for Associative.

Consider ∗ is associative, then

(a ∗ b)* c = a * (b * c)

LHS = (a ∗ b) * c = (ab/2) * c

$= \frac{\frac{ab}{2}\times c}{2}$

= abc/4

RHS = a * (b * c) = a * (bc/2)

$= \frac{a\times\frac{bc}{2}}{2}$

= abc/4

This implies LHS = RHS

Therefore, ∗is associative binary operation.

(iv) On Z^+, define a ∗ b = 2^{ab}

Step 1: Check for commutative

Consider ∗ is commutative, then

a ∗ b = b * a

Which means, $2^{ab} = 2^{ba}$

or $2^{ab} = 2^{ab}$ (which is true)

a ∗ b = b * a for all a, b ∈Z^+

Therefore, ∗is commutative.

Step 2: Check for Associative.

Consider ∗ is associative, then

(a ∗ b)* c = a * (b * c)

LHS = (a ∗ b) * c = (2^{ab}) * c

$=2^{2^{ab}\, c}$

RHS = a * (b * c) = a * 2^{bc}

$= 2^{2^{bc}\, a}$

This implies LHS ≠ RHS

Therefore, ∗is not associative binary operation.

(v) On Z^+, define a ∗ b = a^b

Step 1: Check for commutative

Consider ∗ is commutative, then

a ∗ b = b * a

Which means, $a^b = b^a$

Which is not true

a ∗ b = b * a for all a, b ∈Z^+

Therefore, ∗is not commutative.

Step 2: Check for Associative.

Consider $*$ is associative, then

(a $*$ b)* c = a * (b * c)

LHS = (a^b) * c

= $(a^b)^c$

RHS = a * (b * c) = a * (b^c)

= a^{b^c}

This implies LHS ≠ RHS

Therefore, $*$is not associative.

(vi) On R – {– 1}, define a $*$ b = a/(b+1)

Step 1: Check for commutative

Consider $*$ is commutative, then

a $*$ b = b * a

Which means, a/(b+1) = b/(a+1)

Which is not true

Therefore, $*$is commutative.

Step 2: Check for Associative.

Consider $*$ is associative, then

(a $*$ b)* c = a * (b * c)

LHS = (a $*$ b) * c = (a/(b+1)) * c

= $\frac{\frac{a}{b+1}}{c}$

= a/(c(b+1)

RHS = a * (b * c) = a * (b/(c + 1))

$= \frac{\frac{a}{b}}{c+1}$

= a(c+1)/b

This implies LHS ≠ RHS

Therefore, ∗is not associative binary operation.

3. Consider the binary operation ∧ on the set {1, 2, 3, 4, 5} defined by a ∧ b = min {a, b}. Write the operation table of the operation ∧ .

Solution:

The binary operation ∧ on the set, say A = {1, 2, 3, 4, 5} defined by a ∧ b = min {a, b}. the operation table of the operation ∧ as follow:

^	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	3	3	3
4	1	2	3	4	4
5	1	2	3	4	5

4. Consider a binary operation ∗ on the set {1, 2, 3, 4, 5} given by the following multiplication table (Table 1.2).

(i) Compute (2 ∗ 3) ∗ 4 and 2 ∗ (3 ∗ 4) (ii)
Is ∗ commutative?
(iii) Compute (2 ∗ 3) ∗ (4 ∗ 5).
(Hint: use the following table)

Table 1.2

*	1	2	3	4	5
1	1	1	1	1	1
2	1	2	1	2	1
3	1	1	3	1	1
4	1	2	1	4	1
5	1	1	1	1	5

Solution:

(i) Compute (2 ∗ 3) ∗ 4 and 2 ∗ (3 ∗ 4)

From table: (2 ∗ 3) = 1 and (3 ∗ 4) = 1
(2 ∗ 3) ∗ 4 = 1 * 4 = 1 and

2 ∗ (3 ∗ 4) = 2 * 1 = 1

(ii) Is ∗ commutative?

Consider 2 * 3, we have 2 * 3 = 1 and 3 * 2 = 1

Therefore, * is commutative.

(iii) Compute (2 ∗ 3) ∗ (4 ∗ 5).

From table: (2 ∗ 3) = 1 and(4 ∗ 5) = 1

So (2 ∗ 3) ∗ (4 ∗ 5) = 1 * 1 = 1

5. Let ∗′ be the binary operation on the set {1, 2, 3, 4, 5} defined by a ∗′ b = H.C.F. of a and b. Is the operation ∗′ same as the operation ∗defined in Exercise 4 above? Justify your answer.

Solution: Let A = {1, 2, 3, 4, 5} and a ∗′ b H.C.F. of a and b. Plot a table values, we have

*′	1	2	3	4	5
1	1	1	1	1	1
2	1	2	1	2	1
3	1	1	3	1	1
4	1	2	1	4	1
5	1	1	1	1	5

Operation ∗′ same as the operation *.

6. Let ∗ be the binary operation on N given by a ∗ b = L.C.M. of a and b. Find
(i) 5 ∗ 7, 20 ∗ 16 (ii) Is ∗ commutative?
(iii) Is ∗ associative?
(iv) Find the identity of ∗ in N
(v) Which elements of N are invertible for the operation ∗?

Solution:

(i) 5 ∗ 7 = LCM of 5 and 7 = 35

20 ∗ 16 = LCM of 20 and 16 = 80

(ii) Is ∗ commutative?

a ∗ b = L.C.M. of a and b

b ∗ a = L.C.M. of b and a

a ∗ b = b ∗ a

Therefore ∗ is commutative.

(iii) Is ∗ associative?
For a,b, c ∈ N

(a ∗ b) * c = (L.C.M. of a and b) * c = L.C.M. of a, b and c a ∗ (b * c) = a * (L.C.M. of b and c) = L.C.M. of a, b and c

(a ∗ b) * c = a ∗ (b * c)

Therefore, operation ∗ associative.

(iv) Find the identity of ∗ in N
Identity of * in N = 1

because a * 1 = L.C.M. of a and 1 = a

(v) Which elements of N are invertible for the operation ∗?

Only the element 1 in N is invertible for the operation * because 1 * 1/1 = 1

7. Is ∗ defined on the set {1, 2, 3, 4, 5} by a ∗ b = L.C.M. of a and b a binary operation? Justify your answer.

Solution:

The operation ∗ defined on the set {1, 2, 3, 4, 5} by a ∗ b = L.C.M. of a and b
Suppose, a = 2 and b = 3

2 * 3 = L.C.M. of 2 and 3 = 6

But 6 does not belongs to the set A.
Therefore, given operation * is not a binary operation.

8. Let ∗ be the binary operation on N defined by a ∗ b = H.C.F. of a and b. Is ∗ commutative? Is ∗ associative? Does there exist identity for this binary operation on N?

Solution:

The operation ∗ be the binary operation on N defined by a ∗ b = H.C.F. of a and b

a * b = H.C.F. of a and b = H.C.F. of b and a = b * a

Therefore, operation * is commutative.

Again, (a *b)*c = (HCF of a and b) * c = HCF of (HCF of a and b) and c = a * (b *c) (a *b)*c = a * (b *c)

Therefore, the operation is associative.

Now, 1 * a = a * 1 ≠ a

Therefore, there does not exist any identity element.

9. Let ∗ be a binary operation on the set Q of rational numbers as follows:
(i) a ∗ b = a – b
(ii) a ∗ b = $a^2 + b^2$
(iii) a ∗ b = a + ab
(iv) a ∗ b = $(a – b)^2$
(v) a ∗ b = ab/4
(vi) a ∗ b = ab^2

Find which of the binary operations are commutative and which are associative.

Solution: (i) a
∗ b = a – b

a ∗ b = a – b = - (b – a) = -b * c ≠ b * a (Not commutative)

(a * b) * c = (a – b) * c = (a – (b – c) = a – b + c ≠ a * (b *c) (Not associative)

(ii) a ∗ b = $a^2 + b^2$

a ∗ b = $a^2 + b^2 = b^2 + a^2$ = b * a (operation is commutative)

Check for associative:

(a * b) * c = $(a^2 + b^2) * c^2 = (a^2 + b^2) + c^2$

a * (b *c) = a * $(b^2 + c^2) = a^2 * (b^2 + c^2)^2$

(a * b) * c ≠ a * (b *c) (Not associative)

(iii) a ∗ b = a + ab

a ∗ b = a + ab = a(1 + b)

b * a = b + ba = b (1+a)

a ⁎ b ≠ b * a

The operation * is not commutative

Check for associative:

(a * b) * c = (a + ab) * c = (a + ab) + (a + ab)c

a * (b *c) = a * (b + bc) = a + a(b + bc)

(a * b) * c ≠ a * (b *c)

The operation * is not associative

(iv) a ⁎ b = (a – b)^2

a ⁎ b = (a – b)^2

b * a = (b – a)^2

a ⁎ b = b * a

The operation * is commutative.

Check for associative:

(a * b) * c = (a - b)^2 * c = ((a – b)^2 – c)^2

a * (b *c) = a * (b - c)^2 = (a - (b - c)^2)^2

(a * b) * c ≠ a * (b *c)

The operation * is not associative

(v) a ⁎ b = ab/4

b * a = ba/2 = ab/2

a ⁎ b = b * a

The operation * is commutative.

Check for associative:

(a * b) * c = ab/4 * c = abc/16

a * (b *c) = a * (bc/4) = abc/16

(a * b) * c = a * (b *c)

The operation * is associative.

(vi) a ∗ b = ab^2
b ∗ a = ba^2

a ∗ b ≠ b ∗ a

The operation * is not commutative.

Check for associative:

(a * b) * c = (ab^2) * c = ab^2 c^2

a * (b *c) = a * (b c^2) = ab^2 c^4

(a * b) * c ≠ a * (b *c)

The operation * is not associative.
10. Find which of the operations given above has identity.

Solution: Let I be the identity.

(i)   a * I = a – I ≠ a

(ii) a * I = a^2 – I^2 ≠ a

(iii) a * I = a + a I ≠ a

(iv) a * I = (a – I)^2 ≠ a

(v) a * I = aI/4 ≠ a

Which is only possible at I = 4 i.e. a * I = aI/4 = a(4)/4 = a

(vi) a * I = a $I^2 \neq$ a

Above identities does not have identity element except (V) at b = 4.

11. Let A = N × N and ∗ be the binary operation on A defined by
(a, b) ∗ (c, d) = (a + c, b + d)
Show that ∗ is commutative and associative. Find the identity element for ∗ on A, if any.

Solution: A = N x N and * is a binary operation defined on A. (a, b) * (c, d) = (a + c, b + d)

(c, d) * (a, b) = (c + a, d + b) = (a + c, b + d)

The operation * is commutative

Again, ((a, b) * (c, d)) * (e, f) = (a + c, b + d) * (e, f)
= (a + c + e, b + d + f)

(a, b) * ((c, d)) * (e, f)) = (a, b) * (c+e, e+f) = (a+c+e, b+d+f)

=> ((a, b) * (c, d)) * (e, f) = (a, b) * ((c, d)) * (e, f))

The operation * is associative.

Let (e, f) be the identity function, then

(a, b) * (e, f) = (a + e, b + f)
For identity function, a = a + e => e = 0 and b = b + f => f = 0

As zero is not a part of set of natural numbers. So identity function does not exist.

As 0 ∉ N, therefore, identity-element does not exist.

12. State whether the following statements are true or false. Justify. (i) For an arbitrary binary operation * on a set N, a * a = a ∀ a ∈ N.
(ii) If * is a commutative binary operation on N, then a * (b * c) = (c * b) * a

Solution:
(i) Given: * being a binary operation on N, is defined as a * a = a ∀ a ∈ N

Here operation * is not defined, therefore, the given statement is not true.

(ii) Operation * being a binary operation on N.

c * b = b * c

(c * b) * a = (b * c) * a = a * (b * c)

Thus, a * (b * c) = (c * b) * a, therefore the given statement is true.

13. Consider a binary operation ∗ on N defined as a ∗ b = $a^3 + b^3$. Choose the correct answer.
(A) Is ∗ both associative and commutative?
(B) Is ∗ commutative but not associative? (C) Is ∗ associative but not commutative?
(D) Is ∗ neither commutative nor associative?
Solution:
A binary operation ∗ on N defined as a ∗ b = $a^3 + b^3$,

Also, a ∗ b = $a^3 + b^3 = b^3 + a^3$ = b * a The operation * is commutative.

Again, (a ∗ b)*c = $(a^3 + b^3) * c = (a^3 + b^3)^3 + c^3$

a * (b * c) = $a * (b^3 + c^3) = a^3 + (b^3 + c^3)^3$

⇨ (a ∗ b)*c ≠ a * (b * c)

The operation * is not associative.

Therefore, option (B) is correct.

Miscellaneous Exercise

1. Let f : R → R be defined as f(x) = 10x + 7. Find the function g : R → R such that g o f = f o g = I_R.

Solution:

Firstly, Find the inverse of f.
Let say, g is inverse of f and y
= f(x) = 10x + 7

y = 10x + 7

or x = (y-7)/10

or g(y) = (y-7)/10; where g : Y → N

Now, gof = g(f(x)) = g(10x + 7)

$= \frac{(10x+7)-7}{10}$

= x
=
I_R

Again, fog = f(g(x)) = f((y-7)/10)

= 10((y-7)/10) + 7

= y – 7 + 7 = y

= I_R

Since g o f = f o g = I_R. f is invertible, and

Inverse of f is x = g(y) = (y-7)/10

2. Let f : W → W be defined as f(n) = n – 1, if n is odd and f(n) = n + 1, if n is even. Show that f is invertible. Find the inverse of f. Here, W is the set of all whole numbers.

Solution:

f : W → W be defined as f(n) = n – 1, if n is odd and f(n) = n + 1, if n is even.

Function can be defined as:

$$f(n)=\begin{cases} n-1, & \text{if } n \text{ is odd} \\ n+1, & \text{if } n \text{ is even} \end{cases}$$

f is invertible, if f is one-one and onto.

For one-one:

There are 3 cases:
for any n and m two real numbers:

Case 1: n and m : both are odd

f(n) = n + 1 f(m)
= m + 1
If f(n) = f(m)
=> n + 1 = m + 1
=> n = m

Case 2: n and m : both are even

f(n) = n - 1
f(m) = m - 1
If f(n) = f(m)
=> n - 1 = m - 1
=> n = m

Case 3: n is odd and m is even

f(n) = n + 1 f(m)
= m - 1
If f(n) = f(m)
=> n + 1 = m - 1
=> m - n = 2 (not true, because Even – Odd ≠ Even)

Therefore, f is one-one

Check for onto:

$$f(n)=\begin{cases} n-1, & \text{if } n \text{ is odd} \\ n+1, & \text{if } n \text{ is even} \end{cases}$$

Say f(n) = y, and y ∈ W

Case 1: if n = odd

f(n) = n – 1

n = y + 1
Which show, if n is odd, y is even number.

Case 2: If n is even

f(n) = n + 1

y = n + 1

or n = y – 1
If n is even, then y is odd.

In any of the cases y and n are whole numbers.

This shows, f is onto.

Again, For inverse of f

f^{-1} : y = n – 1

or n = y + 1 and y = n + 1

⇨ n = y – 1

$$f^{-1}(n)=\begin{cases} n-1, & \text{if n is odd} \\ n+1, & \text{if n is even} \end{cases}$$

Therefore, $f^{-1}(y) = y$. This show inverse of f is f itself.

3. If f : R → R is defined by f(x) = $x^2 - 3x + 2$, find f (f(x)).

Solution:
Given: $f(x) = x^2 - 3x + 2$

$f(f(x)) = f(x^2 - 3x + 2)$

$= (x^2 - 3x + 2)^2 - 3(x^2 - 3x + 2) + 2$

$= x^4 - 6x^3 + 10x^2 - 3x$

4. Show that the function f : R → {x ∈ R : – 1 < x < 1} defined by f(x) = $\frac{x}{1+|x|}$, x ∈ R is one one and onto function.

Solution:

The function f : R → {x ∈ R : – 1 < x < 1} defined by f(x) = $\frac{x}{1+|x|}$, x ∈ R **For one-one:**

Say x, y ∈ R
As per definition of |x|;

$$|x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

So f(x) = $\begin{cases} \frac{x}{1-x}, & x < 0 \\ \frac{x}{1+x}, & x \geq 0 \end{cases}$

For x ≥ 0

f(x) = x/(1+x)

f(y) = y/(1+y)

If f(x) = f(y), then

x/(1+x) = y/(1+y)

x(1 + y) = y (1+x)

⇨ x= y

For x < 0

f(x) = x/(1-x)

f(y) = y/(1-y)

If f(x) = f(y), then

x/(1-x) = y/(1-y)

x(1 - y) = y (1- x)

⇨ x= y

In both the conditions, x = y.

Therefore, f is one-one.

Again for onto:

$$f(x) = \begin{cases} \frac{x}{1-x}, & x < 0 \\ \frac{x}{1+x}, & x \geq 0 \end{cases}$$

For x < 0

y = f(x) = x/(1-x)

y(1-x) = x

or x(1+y) = y

or $x = y/(1+y)$ …(1)

For x ≥ 0

$y = f(x) = x/(1+x)$

$y(1+x) = x$

or $x = y/(1-y)$ …(2)

Now we have two different values of x from both the case.

Since $y \in \{x \in R : -1 < x < 1\}$ The value of y lies between -1 to 1.

If y = 1

$x = y/(1-y)$ (not defined)

If y = -1

$x = y/(1+y)$ (not defined)

So x is defined for all the values of y, and $x \in R$

This shows that, f is onto.

Answer: f is one-one and onto.

5. Show that the function f : R → R given by $f(x) = x^3$ is injective.

Solution:

The function f : R → R given by $f(x) = x^3$
Let x , y ∈ R such that f(x) = f(y)

This implies , $x^3 = y^3$

x = y f is one-one. So f is
injective.

6. Give examples of two functions f : N → Z and g : Z → Z such that g o f is injective but g is not injective.
(Hint : Consider f(x) = x and g (x) = | x |)

Solution:
Given: two functions are f : N → Z and g : Z → Z

Let us say, f(x) = x and g(x) = x

gof = (gof)(x) = f(f(x)) = g(x)

Here gof is injective but g is not.
Let us take a example to show that g is not injective: Since g(x) = |x| g(-1) = |-1| = 1 and g(1) = |1| = 1
But -1 ≠ 1

7. Give examples of two functions f : N → Z and g : Z → Z such that g o f is injective but g is not injective.

(Hint : Consider f(x) = x +1 and g (x) = $\begin{cases} x-1 & if\ \ x>1 \\ 1 & if\ \ x=1 \end{cases}$)

Solution:

Given: Two functions f : N → Z and g : Z → Z

Say f(x) = x+ 1

And g (x) = $\begin{cases} x-1 & if\ \ x>1 \\ 1 & if\ \ x=1 \end{cases}$

Check if f is onto:

f : N →N be f(x) = x + 1

say y = x + 1

or x = y – 1

for y = 1, x = 0, does not belong to N

Therefore, f is not onto.

Find gof

For x = 1; gof = g(x + 1) = 1 (since g(x) = 1)
For x > 1 ; gof = g (x + 1) = (x + 1) – 1 = x (since g(x) = x – 1)

So we have two values for gof.

As gof is a natural number, as y = x. x is also a natural number. Hence gof is onto.

8. Given a non empty set X, consider P(X) which is the set of all subsets of X.

Define the relation R in P(X) as follows:

For subsets A, B in P(X), ARB if and only if A ⊂ B. Is R an equivalence relation on P(X)? Justify your answer.

Solution:

A ⊂ A ∴ R is reflexive.

A ⊂ B ≠ B ⊂ A ∴ R is not commutative.

If A ⊂ B, B ⊂ C, then A ⊂ C ∴ R is transitive

Therefore, R is not equivalent relation

9. Given a non-empty set X, consider the binary operation * : P(X) × P(X) → P(X) given by A * B = A ∩ B ∀ A, B in P(X), where P(X) is the power set of X. Show that X is the identity element for this operation and X is the only invertible element in P(X) with respect to the operation * .

Solution:

Let T be a non-empty set and P(T) be its power set. Let any two subsets A and B of T.

A ∪ B ⊂ T

So, A ∪ B ∈ P(T)

Therefore, ∪ is an binary operation on P(T).

Similarly, if A, B $\in$ P(T) and A – B $\in$ P(T), then the intersection of sets and difference of sets are also binary operation on P(T) and A $\cap$ T = A = T $\cap$ A for every subset A of sets

A $\cap$ T = A = T $\cap$ A for all A $\in$ P(T)

T is the identity element for intersection on P(T).

10. Find the number of all onto functions from the set {1, 2, 3, ……….., n} to itself.

Solution: The number of onto functions that can be defined from a finite set A containing n elements onto a finite set B containing elements = 2^n - n.

11. Let S = {a, b, c} and T = {1, 2, 3}. Find F^{-1} of the following functions F from S to T, if it exists.

(i) F = {(a, 3), (b, 2), (c, 1)} **(ii) F = {(a, 2), (b, 1), (c, 1)}**

Solution: (i) F = {(a, 3), (b, 2), (c, 1)}

F(a) = 3, F(b) = 2 and F(c) = 1

F^{-1} (3) = a, F^{-1} (2) = b and F^{-1} (1) = c

F^{-1} = {(3, a), (2, b), (1, c)}

(ii) F = {(a, 2), (b, 1), (c, 1)}

Since element b and c have the same image 1 i.e. (b, 1), (c, 1).

Therefore, F is not one-one function.

12. Consider the binary operations $*$: R × R → R and o : R × R → R defined as a $*$ b = |a – b| and a o b = a, $\forall$ a, b $\in$ R. Show that $*$ is commutative but not associative, o is associative but not commutative. Further, show that $\forall$ a, b, c $\in$ R, a $*$ (b o c) = (a $*$ b) o (a $*$ c). [If it is so, we say that the operation $*$ distributes over the operation o]. Does o distribute over $*$? Justify your answer.

Solution:

Step 1: **Check for commutative and associative for operation** $*$.

a $*$ b = |a – b| and b $*$ a = |b – a| = (a, b)

Operation * is commutative.

a*(b*c) = a*|b-c| = |a-(b-c)| = |a-b+c| and

(a*b)*c = |a-b|*c = |a-b-c|

Therefore, a*(b*c) ≠ (a*b)*c

Operation * is associative.

Step 2: **Check for commutative and associative for operation** o.

aob = a ∀ a, b ∈ R and boa = b

This implies aob boa

Operation o is not commutative.

Again, a o (b o c) = a o b = a and (aob)oc = aoc = a
Here ao(boc) = (aob)oc

Operation o is associative.
Step 3: Check for the distributive properties

If * is distributive over o then, $a*(boc) = a*b = |a-b|$

RHS:
$(a*b)o(a*b) = (a-b)o(a-c) = |a-b|$
= LHS

And, $ao(b*c) = (aob)*(aob)$

LHS
$ao(b*c) = ao(|b-c|) = a$

RHS
$(aob)*(aob) = a*a = |a-a| = 0$
LHS ≠ RHS

Hence, operation o does not distribute over.

13. Given a non-empty set X, let * : P(X) × P(X) → P(X) be defined as A * B = (A – B) ∪ (B – A), ∀ A, B ∈ P(X). Show that the empty set ϕ is the identity for the operation * and all the elements A of P(X) are invertible with A^{-1} = A. (Hint : (A – ϕ) ∪ (ϕ – A) = A and (A – A) ∪ (A – A) = A * A = ϕ).

Solution: x ∈ P(x)

$\phi * A = (\phi - A) \cup (A - \phi) = \phi \cup A = A$

And

$A * \phi = (A - \phi) \cup (\phi - A) = A \cup \phi = A$

ϕ is the identity element for the operation * on P(x).

Also A*A= $(A - A) \cup (A - A)$

= $\phi \cup \phi = \phi$

Every element A of P(X) is invertible with A^{-1} = A.

14. Define a binary operation * on the set {0, 1, 2, 3, 4, 5} as

a * b = $\begin{cases} \boldsymbol{a+b} & \boldsymbol{if\ a+b<6} \\ \boldsymbol{a+b-6} & \boldsymbol{if\ a+b\geq 0} \end{cases}$

Show that zero is the identity for this operation and each element a ≠ 0 of the set is invertible with 6 – a being the inverse of a.

Solution:

Let x = {0, 1, 2, 3, 4, 5} and operation * is defined as

a * b = $\begin{cases} a+b & if\ a+b<6 \\ a+b-6 & if\ a+b\ \geq 0 \end{cases}$

Let us say, $e \in X$ is the identity for the operation *, if a*e = a = e*a $\forall a \in X$

$\begin{cases} a+b=0=b+a, & \text{if } a+b<6 \\ a+b-6=0=b+a-6, & \text{if } a+b\geq 6 \end{cases}$

That is a = -b or b = 6 – a, which shows a ≠ - b

Since x = {0, 1, 2, 3, 4, 5} and $a, b \in X$

Inverse of an element a ∈ x, a ≠ 0, and a^{-1} = 6 – a.

15. Let A = {– 1, 0, 1, 2}, B = {– 4, – 2, 0, 2} and f, g : A → B be functions defined by f(x) = x^2 – x, x ∈ A and g(x) = 2|x – ½| - 1, x ∈ A. Are f and g equal?

Justify your answer. (Hint: One may note that two functions f : A → B and g : A → B such that f(a) = g (a) ∀ a ∈ A, are called equal functions).

Solution:
Given functions are: f(x) = x^2 – x and g(x) = 2|x – ½| - 1

At x = -1
f(-1) = 1^2 + 1 = 2 and g(-1) = 2|-1 – ½| - 1 = 2

At x = 0
F(0) = 0 and g(0) = 0
At x = 1
F(1) = 0 and g(1) = 0
At x = 2
F(2) = 2 and g(2) = 2

So we can see that, for each a ∈ A , f(a) = g(a)

This implies f and g are equal functions.

16. Let A = {1, 2, 3}. Then number of relations containing (1, 2) and (1, 3) which are reflexive and symmetric but not transitive is

(A) 1 (B) 2 (C) 3 (D) 4
Solution:

Option (A) is correct.

As 1 is reflexive and symmetric but not transitive.

17. Let A = {1, 2, 3}. Then number of equivalence relations containing (1, 2) is

(A) 1 **(B) 2** **(C) 3** **(D) 4**

Solution:

Option (B) is correct.

18. Let f : R → R be the Signum Function defined as

$$f(x)=\begin{cases}1, & x>0\\ 0, & x=0\\ -1, & x<0\end{cases}$$

and g : R → R be the Greatest Integer Function given by g (x) = [x], where [x] is greatest integer less than or equal to x. Then, does fog and gof coincide in (0, 1]?

Solution: Given:

f : R → R be the Signum Function defined as

$$f(x)=\begin{cases}1, & x>0\\ 0, & x=0\\ -1, & x<0\end{cases}$$

and g : R → R be the Greatest Integer Function given by g (x) = [x], where [x] is

greatest integer less than or equal to x.

Now, let say x ∈ (0, 1], then

[x] = 1 if x =1 and

[x] = 0 if 0< x < 1

Therefore:

$$fog(x)=f(g(x))=f([x])$$

$$=\begin{cases}f(1), & \text{if } x=1\\ f(0), & \text{if } x\in(0,1)\end{cases}$$

$$=\begin{cases}1, & \text{if } x=1\\ 0, & \text{if } x\in(0,1)\end{cases}$$

Gof(x) = g(f(x)) = g(1) = [1] = 1

For x > 0

When x ∈ (0, 1), then fog = 0 and gof = 1
But fog (1) ≠ gof (1)

This shows that, fog and gof do not concide in 90, 1].

19. Number of binary operations on the set {a, b} are

(A) 10 (B) 16 (C) 20 (D) 8
Solution:
Option (B) is correct.

A = {a, b} and

A x A = {(a,a), (a,b),(b,b),(b,a)}

Number of elements = 4

So, number of subsets = 2^4 = 16.